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Fork-decomposition of the Cartesian Product of Graphs

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Abstract

Let $G = (V, E)$ be a graph. *Fork* is a tree obtained by subdividing any edge of a star of size three exactly once. In this paper, we investigate the necessary and sufficient condition for the fork-decomposition of Cartesian product of graphs.

Keywords: Decomposition, Fork, Product graph, Cartesian product.
Mathematics Subject Classification (2010): 05C70, 05C51, 05C76.

Introduction

We consider simple, finite, and undirected graphs. Let K_n denote the complete graph on n vertices and $K_{m,n}$ denote the complete bipartite graph with partite sets of sizes m and n. Let P_k denote the path of length $k-1$ and S_k denote the star of size $k-1$. A vertex of degree 1 is called a *pendant vertex* and the vertex adjacent to it is called the *support vertex*. Terms not defined here are used in the sense of Bondy and Murty [4].

Let $L = \{H_1, H_2, \dots H_r\}$ be a family of subgraphs of G. An *L-decomposition* of G is an edge-disjoint decomposition of G into positive integers α_i copies of H_i where $i \in \{1,2,\dots,r\}$. Furthermore, if each $H_i (i \epsilon \{1,2,\dots,r\})$ is isomorphic to a graph H, then we say that G has an *H-decomposition*.

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The obvious necessary condition for the existence of a $\{H_1, H_2, \dots H_r\}$-decomposition of G is $\sum_{i=1}^{r} \alpha_i e(H_i) = e(G)$ (1)

We call this equation as necessary sum condition.

The Fork graph was defined by Simone and Sassano in the name of *chair graph* in 1993, when they studied the stability number of bull and chair-free graphs [5]. A tree with degree sequence (1,1,1,2,3) is unique and is nothing but the fork defined above. Hence the subgraph Fork is also called a chair  or (3,2,1,1,1)-tree.
The decomposition of arbitrary graphs into subgraphs of small size is assuming importance in literature. There are several studies on the isomorphic decomposition of graphs into sunlet [1], cycles [2], trees [3], paths [8, 11], stars [12], etc. In 2013, P. Chithra Devi and J. Paulraj Joseph studied the P_4 Decomposition of Product graphs [6]. The general problem of H-decompositions was proved to be NP-complete for any H of size greater than 2 by Dor and Tarsi [7]. The decomposition of complete bipartite graphs, complete graphs, and corona graphs into Fork was studied in [9]. In this paper, we investigate the decomposition of the Cartesian product of graphs into forks.

Definition 1.1. [10] The Cartesian product of two graphs G and H, denoted by $G\square H$, is the graph whose vertex set is $V(G) \times V(H)$; two vertices (g, h) and (g', h') are adjacent in $G\square H$ precisely if $g = g'$ and $hh' \epsilon E(H)$, or $gg' \epsilon E(G)$ and $h = h'$. The number of edges in $G\square H$ is $|V(G)||E(H)| + |V(H)||E(G)|$.

The following results are used in the subsequent section.

Theorem 1.2. [9] The complete bipartite graph $K_{m,n}$ is fork-decomposition if and only if $mn \equiv 0 (mod\ 4)$ except $K_{2,4i+2}, (i = 1,2, \dots)$.

Theorem 1.3. [9] The Complete graph K_n can be decomposed into forks if and only if $n = 8k$ or $n = 8k + 1$, for all $k \geq 1$.

Theorem 1.4. [9] $C_n \circ \overline{K}_m$ is fork-decomposable if and only if $m = 1$ and $n = 2k$ or $m = 3$.

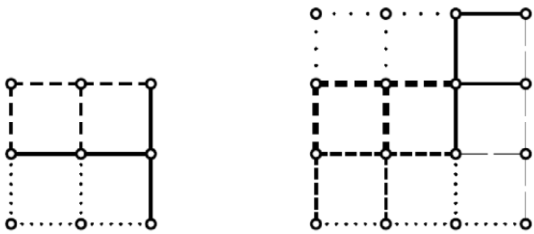
Theorem 1.5. [9]

For $\boldsymbol{m \geq 3, K_m \circ K_1}$ is fork-decomposable if and only if $\boldsymbol{m \equiv 0, 7 \ (mod\ 8)}$
For $m \geq 3, K_m \circ \overline{K}_2$ is fork-decomposable if and only if $m \equiv 0,5 \ (mod\ 8)$

Cartesian product of two Paths
In this section, we give the necessary and sufficient conditions for the decomposition of Cartesian product of two paths into forks. The following three lemmas are used in proving the decomposition of Cartesian product of two paths into forks.

Lemma 2.1. $P_3 \square P_3$ and $P_4 \square P_4$ are fork-decomposable.

Proof. The fork-decomposition is depicted in Figure 2.1.



(a) $P_3 \square P_3$ (b) $P_4 \square P_4$

Figure 2.1 Fork-decomposition of $P_3 \square P_3$ and $P_4 \square P_4$

Lemma 2.2. Let G_1 be the graph obtained by joining one of the vertices of degree 2 of $P_{2k-1} \circ K_1$ $(k \geq 1)$ to the alternate vertices of C_4. Then G_1 is fork-decomposable.

Proof. The graph G_1 is given in Figure 2.2

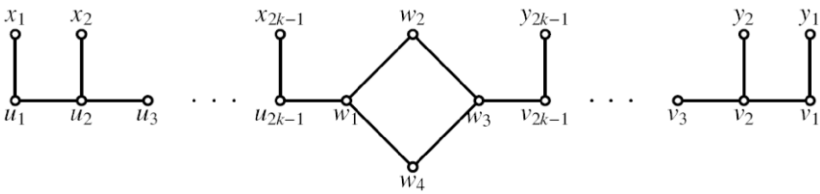


Figure 2.2 The graph G_1

Let $u_1, u_2, \dots, u_{2k-1}$ and $v_1, v_2, \dots, v_{2k-1}$ be the vertices of first and second copy of P_{2k-1} in $P_{2k-1} \circ K_1$ respectively. Let $x_i, y_i (1 \leq i \leq$

$2k-1$) be the pendant vertices of u_i and v_i respectively and let w_1, w_2, w_3, w_4 be the vertices of C_4.

Then a fork-decomposition of G_1 is given by $\{u_i x_i, u_i u_{i+1}, u_{i+1} x_{i+1}, u_{i+1} u_{i+2}\}$, $\{v_i y_i, v_i v_{i+1}, v_{i+1} y_{i+1}, v_{i+1} y_{i+2}\}$, $\{u_{2k-1} x_{2k-1}, u_{2k-1} w_1, w_1 w_2, w_1 w_4\}$, $\{v_{2k-1} y_{2k-1}, v_{2k-1} w_3, w_3 w_2, w_3 w_4\}$ for $i = 1,3,\ldots,2k-3$.

Lemma 2.3. Let G_2 be the graph obtained by identifying an end copy of $P_3 \square P_3$ of $P_3 \square P_{2k+2}$ with another end copy of $P_3 \square P_3$ of $P_3 \square P_{2k+2}$ and deleting the edges of third copy of P_{2k} in each of the graphs $P_3 \square P_{2k+2}$ as shown in Figure 2.3. Then G_2 is fork-decomposable.

Proof. Let us label the vertices of G_2 as given in Figure 2.3

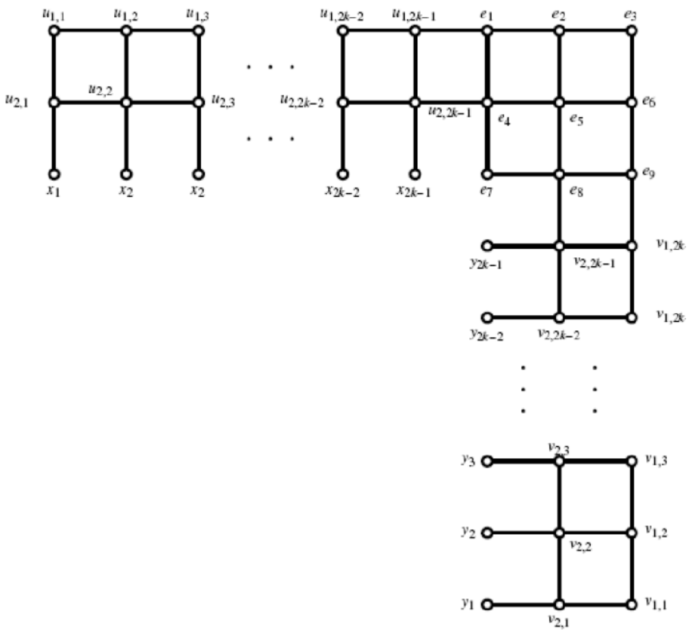


Figure 2.3 The graph G_2

The fork-decomposition of G_2 is given by $\{x_i u_{2,i}, u_{2,i} u_{1,i}, u_{2,i} u_{2,i+1}, u_{1,i} u_{1,i+1}\}$, $\{y_i v_{2,i}, v_{2,i} v_{1,i}, v_{2,i} v_{2,i+1}, v_{1,i} v_{1,i+1}\}$, $\{x_{2k-1} u_{2,2k-1}, u_{2,2k-1} u_{1,2k-1}, u_{2,2k-1} e_4, u_{1,2k-1} e_1\}$, $\{y_{2k-1} v_{2,2k-1}, v_{2,2k-1} v_{1,2k-1}, v_{2,2k-1} e_8, v_{1,2k-1} e_9\}$, $\{e_1 e_2, e_1 e_4, e_2 e_5, e_2 e_3\}$, $\{e_3 e_6, e_6 e_9, e_5 e_6, e_8 e_9\}$, $\{e_4 e_7, e_5 e_8, e_7 e_8, e_8 e_9\}$ for $i = 1$ to $2k-2$.

In the following theorem, we give necessary and sufficient conditions for the fork-decomposition of Cartesian product of paths.

Theorem 2.4. $P_n \square P_m$ is fork-decomposable if and only if $m \equiv n \ (mod\ 4)$ where $n \leq m$.

Proof. If $P_n \square P_m$ is fork-decomposable, then the total number of edges in $P_n \square P_m$, given as $2mn - (m+n)$ must be a multiple of four. Since $2mn$ is even, m and n must be both even or both odd. Hence the difference between m and n must be even. Thus $2mn - m - n = 4k$ implies $m - n = 4k - 2mn + 2m = 4k - 2m(n-1)$, where k is a positive integer. Since m and n are both even or both odd, $m(n-1)$ must be even. That is, $m(n-1) = 2l$ where l is a positive integer. Then $m - n = 4k - 4l = 4(k-l)$. Hence, $m \equiv n \ (mod\ 4)$.

Conversely, assume that $m \equiv n \ (mod\ 4)$. We shall prove that $P_n \square P_m$ is fork-decomposable, by considering two cases.

Case i. $n = m$

If n is even, let $n = 2i$. The proof is by induction on n. Since $P_4 \square P_4$ is fork-decomposable by Lemma 2.1, the result is true for $n = 4$. Assume that the result is true for $n = 2i - 2$. Since $P_{2i} \square P_{2i}$ can be decomposed into $P_{2i-2} \square P_{2i-2}$ and G_2, by induction $P_{2i-2} \square P_{2i-2}$ is fork-decomposable and by Lemma 2.3, G_2 is fork-decomposable. Hence $P_{2i} \square P_{2i}$ is fork-decomposable.

If n is odd, let $n = 2i + 1$. Since $P_3 \square P_3$ is fork-decomposable by Lemma 2.1, the result is true for $n = 3$. $P_{2i+1} \square P_{2i+1}$ can be decomposed into $P_{2i} \square P_{2i}$ and G_1 . Since $P_{2i} \square P_{2i}$ is fork-decomposable and by Lemma 2.2, G_1 is fork-decomposable, $P_{2i+1} \square P_{2i+1}$ is fork-decomposable. Hence $P_n \square P_n$ is fork-decomposable.

Case ii. $n \neq m$

If $n = 2$, $m = 2 + 4a$, for some positive integer $a \geq 1$. For $a = 1$, the graph $P_2 \square P_6$ is fork-decomposable as shown in Figure 2.4.

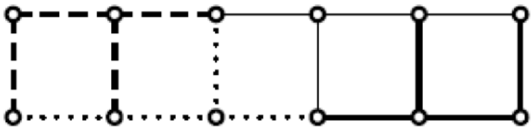


Figure 2.4 $P_2\square P_6$

Consider the graph H_1 obtained from $P_2\square P_4$ with pendant edge attached to first copy of P_2. The graph H_1 is fork-decomposable as shown in the Figure 2.5

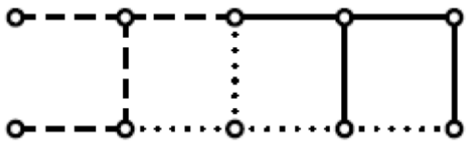


Figure 2.5 The graph H_1

For $a>1$, the graph $P_2\square P_{2+4a}$ can be decomposed into one copy of $P_2\square P_6$ and $a-1$ copies of H_1 and hence $P_2\times P_m$ is fork-decomposable for $m=2+4a$.

If $n\neq 2$, consider the graph H_2 obtained by removing $P_n\square P_n$ from $P_n\square P_m$. Let $l=m-n$. Clearly, l is a multiple of 4 . Let $\{u_{1,i},u_{i,2},\dots,u_{i,n}\}$ be the vertices of $P_l\square P_n$ in H_2 where $i=1$ to l. Let $u_{0,1},u_{0,2},\dots,u_{0,n}$ be the pendant vertices attached to $u_{1,1},u_{1,2},\dots,u_{1,n}$ respectively.

Then a fork-decomposition of H_2 is given by $\{u_{i,1}u_{i,2},u_{i,1}u_{i-1,1},u_{i,2}u_{i-1,2},u_{i,1}u_{i+1,1}\}$, $\{u_{i+1,2}u_{i+1,1},u_{i+1,2}u_{i,2},u_{i+1,2}u_{i+2,2},u_{i+2,2}u_{i+3,2}\}$, $\{u_{i+2,1}u_{i+2,2},u_{i+2,1}u_{i+1,1},u_{i+2,1}u_{i+3,1},u_{i+3,1}u_{i+3,2}\}$ for $i=1,5,9,\dots,m-3$ and $\{u_{j,k}u_{j,k-1},u_{j,k}u_{j-1,k},u_{j,k}u_{j+1,k},u_{j+1,k}u_{j+1,k-1}\}$ for $j=1,3,\dots,m-1$ and $k=3,4,\dots,n$.

Since $P_n\square P_m$ can be decomposed into $P_n\square P_n$ and H_2 which are fork-decomposable, $P_n\square P_m$ is fork-decomposable.

Cartesian Product of path and cycles

In this section, we give the necessary and sufficient conditions for the decomposition of Cartesian product of path and cycles into forks.

The following lemma is used in proving the necessary and sufficient conditions for the fork-decomposition of Cartesian product of paths.

Lemma 3.1. The graph $P_2\square C_n$ is fork-decomposable if and only if $n \equiv 0\ (mod\ 4)$.

Proof. If $P_2\square C_n$ is fork-decomposable, then $|E(P_2\square C_n)| = 3n \equiv 0\ (mod\ 4)$, and hence $n \equiv 0\ (mod\ 4)$.

Conversely, assume that $n \equiv 0\ (mod\ 4)$.

Let $V(C_n) = \{w_1, w_2, \dots, w_n\}$ and $V(P_2) = \{x_1, x_2\}$.

Then $V(P_2\square C_n) = \{(x_1, w_1), (x_1, w_2), \dots, (x_1, w_n), (x_2, w_1), (x_2, w_2), \dots, (x_2, w_n)\}$.

Rename the following vertices: $(x_1, w_j) = u_j$, $(x_2, w_k) = v_k$, $1 \le j$, $k \le n$.

Then a fork-decomposition of $P_2\square C_n$ is given by $\{u_iv_i, u_iu_{i+1}, u_iu_{i-1}, u_{i-1}v_{i-1}\}$, $\{v_{i+1}u_{i+1}, v_{i+1}v_{i-1}, v_{i+1}v_{i+2}, v_{i-1}v_{i-2}\}$, $\{u_{i+2}v_{i+2}, u_{i+2}u_{i+1}, u_{i+2}u_{i+3}, v_{i+2}v_{i+3}\}$ for $i \equiv 2\ (mod\ 4)$. The subscripts are taken modulo n.

Theorem 3.2. $C_m\square P_n$ is fork-decomposable if and only if $m = 4k$.

Proof. Let $G = C_m\square P_n$. If G is fork-decomposable, then $|E(C_m\square P_n)| \equiv 0\ (mod\ 4)$ which implies $2mn - m = 4k$. Thus $m(2n-1) = 4k$. Since $2n-1$ is odd, m must be a multiple of 4.

Conversely, assume that $m = 4k$. Let $H = C_m\square P_2$. Then $C_m\square P_2$ can be decomposed into H and $G - H$. By Lemma 3.1, H can be decomposed into forks and $G - H$ can be decomposed into $n-2$ copies of $C_m \circ K_1$. Since m is a multiple of 4, by Theorem 1.4, $C_m \circ K_1$ can be decomposed into forks and hence G is fork-decomposable.

The following lemma is used in proving the existence of necessary and sufficient conditions for the fork-decomposition of Cartesian product of cycles.

Lemma 3.3. Let G be the graph obtained from $P_2\square C_n$ with pendant vertex attached to the first vertex of each copy of P_2. Then G is fork-decomposable for all $n \ge 3$.

Proof. Let $V(C_n) = \{x_1, x_2, \dots, x_n\}$ and $V(K_2) = \{y_1, y_2\}$.

Then $V(P_2 \square C_n) = \{(x_1, y_1), (x_2, y_1), \ldots, (x_n, y_1), (x_1, y_2), (x_2, y_2), \ldots, (x_n, y_2)\}$.

Rename the vertices $(x_i, y_1) = u_i$ and $(x_i, y_2) = v_i$ for all $1 \leq i \leq n$. Let w_i be the pendant vertex attached to each u_i.

Then a fork-decomposition of G is given by $\{u_n u_1, u_n w_n, u_n v_n, v_n v_1\}$ and $\{u_i u_{i+1}, u_i w_i, u_i v_i, v_i v_{i+1}\}$ for $1 \leq i \leq n-1$. The subscripts are taken modulo n.

Theorem 3.4. $C_m \square C_n$ is fork-decomposable if and only if either m or n is even.

Proof. If $C_m \square C_n$ is fork-decomposable, then $|E(C_m \square C_n)| \equiv 0 \ (mod\ 4)$, implies $2mn = 4k$. Hence either m or n is even.

Conversely, assume that n is even and let $G = C_m \square C_n$. Then G can be decomposed into $\frac{n}{2}$ copies of $C_m \square K_2$ with pendant edge attached to the first copy of C_m which in turn can be decomposed into forks by Lemma 3.3. Similarly, we can prove the result for even values of m.

Cartesian Product of centipede and path

In this section, we give necessary and sufficient conditions for the Cartesian product of centipede and path into forks.

Theorem 4.1. Let G be an m-centipede $(P_m \circ K_1)$ and if $m \equiv 0 (mod\ 2)$, then $G \square P_n$ is fork-decomposable for all $n \equiv 0 (mod\ 4)$.

Proof. Let G be an m-centipede. Let $V(P_m) = \{x_1, x_2, \ldots, x_m\}$ and let y_i be the pendant vertex adjacent to $x_i (1 \leq i \leq m)$. Let $\{x_{i1}, x_{i2}, \ldots, x_{in}, (1 \leq i \leq m)\}$ be the vertices of a path in n copies of m-centipede and let $\{y_{i1}, y_{i2}, \ldots, y_{in}\}$ be the pendant vertices in n copies of m-centipede attached to $x_{ij} (1 \leq i \leq m$ and $1 \leq j \leq n)$ respectively.

The set of forks F_1 and F_2 can be obtained as $\{x_{ij} x_{i(j+1)}, x_{ij} x_{(i+1)j}, x_{ij} y_{ij}, y_{ij} y_{(i+1)j}\}$ for $1 \leq i \leq n-1$ and $1 \leq j \leq m-1$ and $\{x_{ij} x_{i(j+1)}, x_{ij} y_{ij}, x_{ij} x_{i(j-1)}, x_{i(j-1)} y_{i(j-1)}\}$ for $j = 2,4, \ldots, m-2$ and $i = n$. If we remove F_1 and F_2, we get a component M obtained by adding an edge to the end vertex $y_{(n-1)m}$ of path in $P_2 \square P_{n-1}$.

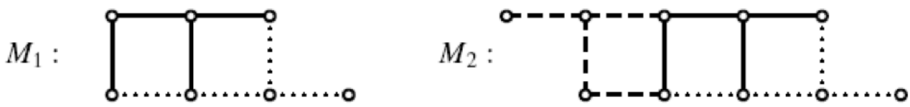


Figure 4.1 The graphs M_1 and M_2

This component M can be decomposed into component M_1 which is obtained by adding an edge to the end vertex of P_3 in $P_2 \square P_3$ and $\frac{n}{4} - 1$ copies of M_2 obtained by adding an edge to the first vertex of the first copy and end vertex of the second copy of P_4 in $P_2 \square P_4$ respectively. The decomposition of M_1 and M_2 are shown in the Figure 4.1.

Cartesian Product of complete graph and path

In the following theorem, we give necessary and sufficient conditions for the fork-decomposition of Cartesian product of complete graph and path.

Theorem 5.1. The graph $K_m \square P_n$ is fork-decomposable if and only if it satisfies any one of the following conditions.

1. $m \equiv 0 \ (mod\ 8)$
2. $n \equiv 0 \ (mod\ 2)$ and $m \equiv 0 \ (mod\ 4)$
3. $n \equiv 2 \ (mod\ 4)$ and $m \equiv 2 \ (mod\ 4)$
4. $n \equiv 3 \ (mod\ 4)$ and $m \equiv 5 \ (mod\ 8)$
5. $n \equiv 1 \ (mod\ 4)$ and $m \equiv 1 \ (mod\ 8)$

Proof. If $K_m \square P_n$ is fork-decomposable, then total number of edges in $K_m \square P_n$ is $m(n-1) + n\left(\frac{m(m-1)}{2}\right) = \frac{m}{2}(n(m+1)-2) \equiv 0 \ (mod\ 4)$. Then $m(n(m+1)-2) = 8k$. Clearly, $m \equiv 0 \ (mod\ 8)$ which is condition (1).

Suppose $m \equiv 1 \ (mod\ 8)$, then $(8k+1)(n(8k+1+1)-2) \equiv 0 \ (mod\ 8)$, which implies $n(8k+2) - 2 \equiv 0 \ (mod\ 8)$. Then, $n(4k+1) - 1 \equiv 0 \ (mod\ 4)$ which implies $n - 1 \equiv 0 \ (mod\ 4)$. Thus $n \equiv 1 \ (mod\ 4)$ which is condition (5).

Suppose $m \equiv 2 \ (mod\ 8)$, then $(8k+2)(n(8k+2+1)-2) \equiv 0 \ (mod\ 8)$, which implies $2n(8k+3) - 4 \equiv 0 \ (mod\ 8)$. Then $n(8k +$

$3) - 2 \equiv 0\ (mod\ 4)$ which implies $3n - 2 \equiv 0\ (mod\ 4)$. Thus $n \equiv 2\ (mod\ 4)$ which is condition (3).

Suppose $m \equiv 3\ (mod\ 8)$, then $(8k+3)(n(8k+3+1)-2) \equiv 0\ (mod\ 8)$, which implies $3n(8k+4) - 6 \equiv 0\ (mod\ 8)$. Then $12n(2k+1) \equiv 6\ (mod\ 8)$ which implies $6\,n(2k+1) \equiv 3\ (mod\ 4)$. Here n is not a positive integer. Hence this condition does not hold.

Suppose $m \equiv 4\ (mod\ 8)$, then $(8k+4)(n(8k+4+1)-2) \equiv 0\ (mod\ 8)$, which implies $4n(8k+5) - 8 \equiv 0\ (mod\ 8)$. Then $4n(5) \equiv 0\ (mod\ 8)$ which implies $5n \equiv 0\ (mod\ 2)$. Thus $n \equiv 0\ (mod\ 2)$ which is condition (2).

Suppose $m \equiv 5\ (mod\ 8)$, then $(8k+5)(n(8k+5+1)-2) \equiv 0\ (mod\ 8)$, which implies $5n(8k+6) - 10 \equiv 0\ (mod\ 8)$. Then $40nk + 30n - 10 \equiv 0\ (mod\ 8)$ which implies $20nk + 15n - 5 \equiv 0\ (mod\ 4)$. Then $15n - 5 \equiv 0\ (mod\ 4)$ which implies $n - 1 \equiv 0\ (mod\ 4)$. Thus $n \equiv 3\ (mod\ 4)$ which is condition (5).

Suppose $m \equiv 6\ (mod\ 8)$, then $(8k+6)(n(8k+6+1)-2) \equiv 0\ (mod\ 8)$, which implies $6n(8k+7) - 12 \equiv 0\ (mod\ 8)$. Then $42n - 12 \equiv 0\ (mod\ 8)$ which implies $21\ n \equiv 6\ (mod\ 4)$. Then $21\ n \equiv 2\ (mod\ 4)$ which implies $n \equiv 2\ (mod\ 4)$ which is condition (2).

Suppose $m \equiv 7\ (mod\ 8)$, then $(8k+7)(n(8k+7+1)-2) \equiv 0\ (mod\ 8)$, which implies $7n(8k+8) - 14 \equiv 0\ (mod\ 8)$ which is not possible. Hence this condition does not hold.

Now let us prove the converse part in 5 cases.

Case 1. $m \equiv 0\ (mod\ 8)$

The graph $K_m \square P_n$ can be decomposed into $\frac{m}{8}$ copies of K_8, $\frac{(n-1)m}{8}$ copies of $K_8 \circ K_1$ and $n\binom{\frac{m}{8}}{2}$ copies of $K_{8,8}$. By Theorem 1.3, 1.5 and 1.2, K_8, $K_8 \circ K_1$ and $K_{8,8}$ are fork-decomposable. Hence $K_m \square P_n$ is fork-decomposable.

Case 2. $n \equiv 0\ (mod\ 2)$ and $m \equiv 0\ (mod\ 4)$

First let us prove the result for $K_4 \square P_n$. If $n = 2$, let the vertices of first and second copy of K_4 be $\{x_1, x_2, x_3, x_4\}$ and $\{y_1, y_2, y_3, y_4\}$ respectively. Then a fork-decomposition of $K_4 \times P_2$ is given by $\{x_1x_4, x_1x_3, x_1x_2, x_3y_3\}$, $\{x_2x_4, x_2x_3, x_2y_2, y_2y_3\}$, $\{y_1y_2, y_1y_3, y_1x_1, y_3y_4\}$ and $\{y_4y_2, y_4y_1, y_4x_4, x_4x_3\}$.

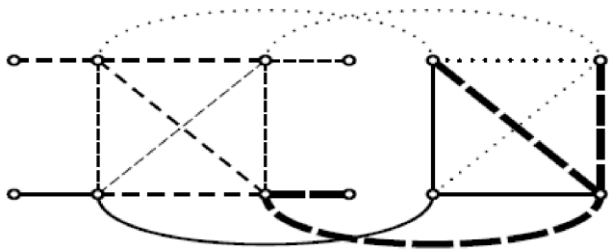


Figure 5.1 The graph A

If $n \geq 4$, the graph $K_4 \Box P_n$ can be decomposed into one copy of $K_4 \Box P_2$ and $\frac{n}{2} - 1$ copies of the graph A given in Figure 5.1. Since the graph A is fork decomposable as shown in the Figure 5.1, $K_4 \Box P_n$ is fork-decomposable.

The graph $K_m \Box P_n$ can be decomposed into $\frac{m}{4}$ copies of $K_4 \Box P_n$ and $n\binom{\frac{m}{4}}{2}$ copies of $K_{4,4}$. By Theorem 1.2, $K_{4,4}$ is fork-decomposable. Hence $K_m \Box P_n$ is fork-decomposable.

Case 3. $n \equiv 2\ (mod\ 4)$ and $m \equiv 2\ (mod\ 4)$.

Firstly, let us prove the result for $m = 6$. The graph $K_6 \Box P_n$ can be decomposed into $K_4 \Box P_n$, $P_n \Box P_2$ and n copies of $K_{2,4}$. By Case (2), $K_4 \Box P_n$ is fork-decomposable. By Theorem 1.2, $K_{2,4}$ is fork-decomposable and by Theorem 2.4, $P_n \Box P_2$ is fork-decomposable. Hence $K_6 \Box P_n$ is fork-decomposable.

For $m > 6$, the graph $K_m \Box P_n$ can be decomposed into $K_6 \Box P_n$, $K_{m-6} \Box P_n$ and $K_{6,m-6}$. By Case (2), $K_{m-6} \Box P_n$ is fork-decomposable and by Theorem 1.2, $K_{6,m-6}$ is fork-decomposable. Hence $K_m \Box P_n$ is fork-decomposable.

Case 4. $n \equiv 3\ (mod\ 4)$ and $m \equiv 5\ (mod\ 8)$.

Firstly, we shall prove that $K_5 \Box P_n$ is fork-decomposable. If $n = 3$, let the vertices of first, second and third copy of K_5 be u_i, v_i, $w_i (1 \leq i \leq 5)$ respectively. Then a fork-decomposition of $K_5 \Box P_3$ is given by

$\{u_1u_2, u_1u_3, u_1u_4, u_4u_5\}$, $\{u_2u_3, u_2u_4, u_2u_5, u_5u_1\}$,
$\{u_3u_4, u_3u_5, u_3v_3, v_3w_3\}$, $\{v_2v_1, v_2v_5, v_2w_2, v_5u_5\}$,
$\{v_3v_2, v_3v_1, v_3v_5, v_1u_1\}$, $\{v_4v_3, v_4v_2, v_4u_4, v_2u_2\}$,

$\{v_1v_5, v_1v_4, v_1w_1, v_4w_4\}$, $\{w_1w_3, w_1w_4, w_1w_5, w_3w_2\}$,
$\{w_2w_1, w_2w_5, w_2w_4, w_4w_3\}$, $\{w_5w_4, w_5w_3, w_5v_5, v_5v_4\}$.

If $n \geq 3$, $K_5 \square P_n$ can be decomposed into $\left\lceil \frac{m}{4} \right\rceil$ copies of $K_5 \square P_3$ and $\left\lceil \frac{m}{4} \right\rceil - 1$ copies of $K_5 \circ \overline{K}_2$. By Theorem 1.5, $K_5 \circ \overline{K}_2$ is fork-decomposable and hence $K_5 \square P_n$ is fork-decomposable.

Now we shall prove the result for m. The graph $K_m \square P_n$ can be decomposed into $K_{m-5} \square P_n$, $K_5 \square P_n$ and m copies of $K_{5,m-5}$. Since $m \equiv 5\ (mod\ 8)$, $K_{m-5} \square P_n$ is fork-decomposable by Case (1) and by Theorem 1.2, $K_{5,m-5}$ is fork-decomposable. Hence $K_m \square P_n$ is fork-decomposable.

Case 5. $n \equiv 1\ (mod\ 4)$ and $m \equiv 1\ (mod\ 8)$.

Firstly, we shall prove that $K_9 \square P_n$ is fork-decomposable. The graph $K_9 \square P_n$ can be decomposed into $K_4 \square P_{n-1}$, $K_5 \square \overline{K}_2$ and a graph H obtained by attaching pendant edges to four consecutive vertices of K_9. By Case (2) and (4), $K_4 \square P_{n-1}$ and $K_5 \square P_{n-2}$ are fork-decomposable respectively. By Theorem 1.5, $K_5 \circ \overline{K}_2$ is fork-decomposable.

Now we have to prove that the graph H is fork-decomposable. Let $\{x_1, x_2, \ldots, x_9\}$ be the vertices of K_9 and let $\{y_1, y_2, y_3, y_4\}$ be the pendant vertices attached to $x_i (i = 1 \text{ to } 4)$ respectively. Then a fork-decomposition of H is given by

$\{x_1x_5, x_1x_6, x_1x_9, x_9x_8\}$, $\{x_2x_4, x_2x_5, x_2x_7, x_5x_8\}$,
$\{x_2x_6, x_2x_8, x_2x_9, x_6x_5\}$, $\{x_2x_3, x_2y_2, x_2x_1, x_1y_1\}$,
$\{x_3x_6, x_3x_8, x_3x_9, x_8x_1\}$, $\{x_4x_8, x_4y_4, x_4x_1, x_1x_3\}$,
$\{x_4x_3, x_4x_9, x_4x_6, x_6x_8\}$, $\{x_5x_4, x_5x_3, x_5x_9, x_9x_7\}$,
$\{x_7x_8, x_7x_5, x_7x_6, x_6x_9\}$, $\{x_7x_4, x_7x_3, x_7x_1, x_3y_3\}$.

Hence $K_9 \times P_n$ is fork-decomposable.

Now we shall prove that the result for m. The graph $K_m \square P_n$ can be decomposed into $K_9 \square P_n$, $K_{m-9} \square P_n$ and n copies of $K_{9,m-9}$. Since $m \equiv 1\ (mod\ 8)$, by Case (1), $K_{m-9} \square P_n$ is fork-decomposable and by Theorem1.2, $K_{9,m-9}$ is fork-decomposable. Hence $K_m \square P_n$ is fork-decomposable.

Cartesian product of complete graph and cycle

In the following theorem, we give necessary and sufficient conditions for the fork-decomposition of Cartesian product of complete graph and cycle.

Theorem 6.1 The graph $K_m \square C_n$ is fork-decomposable if and only if it satisfies any one of the following conditions.

1. n is even and $m \equiv 0(mod\ 4)$ or $m \equiv -1(mod\ 4)$
2. n is odd and $m \equiv 0(mod\ 8)$ or $m \equiv -1(mod\ 8)$
3. $n \equiv 0(mod\ 4)$

Proof: If the graph $K_m \square C_n$ is fork-decomposable, then the total number of edges is $n\frac{m(m-1)}{2} + mn = mn\left(\frac{m+1}{2}\right) = n\left(\frac{m^2+m}{2}\right) \equiv 0(mod\ 4)$. That is $n\big(m(m+1)\big) \equiv 0(mod\ 8)$.

If n is odd, then $m(m+1) \equiv 0(mod\ 8)$. Hence $m \equiv 0(mod\ 8)$ or $m \equiv -1(mod\ 8)$ which is condition 2.

Obviously $m(m+1)$ is even and if n is even, then either $n \equiv 0(mod\ 4)$ which is condition 3 or $m(m+1)$ must be a multiple of 4. That is, $m \equiv 0(mod\ 4)$ or $m \equiv -1(mod\ 4)$, which is condition 1.

Now we shall prove the converse part in 3 cases.

Case 1(a). $n \equiv 0(mod\ 2)$ and $m \equiv 0(mod\ 4)$.
When $m = 4$, the graph $K_4 \square C_n$ can be decomposed into $K_3 \square C_n$ and $C_n \circ \underline{K_3}$. By Theorem 3.4, the graph $K_3 \square C_n$ is fork-decomposable and by Theorem 1.4, $C_n \circ \underline{K_3}$ is fork-decomposable. Hence the graph $K_4 \square C_n$ is fork-decomposable. The graph $K_m \square C_n$ can be decomposed into $\frac{m}{4}$ copies of $K_4 \square C_n$ and $n\binom{\frac{m}{4}}{2}$ copies of $K_{4,4}$. By Theorem 1.2, $K_{4,4}$ is fork-decomposable. Hence $K_m \square P_n$ is fork-decomposable.

Case 1(b). $n \equiv 0(mod\ 2)$ and $m \equiv -1(mod\ 4)$.

When $m = 3$, the graph $K_3 \square C_n$ is fork-decomposable by Theorem 3.4. Consider the graph $K_m \square C_n$. The graph $K_m \square C_n$ can be decomposed into $K_3 \square C_n$, $K_{m-3} \square C_n$ and n copies of $K_{3,m-3}$. By Theorem 1.2, $K_{3,m-3}$ is fork-decomposable and by Case 1(a), $K_{m-3} \square C_n$ is fork-decomposable. Hence $K_m \square C_n$ is fork-decomposable.

Case 2(a). $n \equiv 1(mod\ 2)$ and $m \equiv 0(mod\ 8)$.

When $m = 8$, the graph $K_8 \square C_n$ can be decomposed into n copies of $K_8 \circ K_1$. By Theorem 1.5, $K_8 \circ K_1$ is fork-decomposable. Hence the graph $K_8 \square C_n$ is fork-decomposable. The graph $K_m \square C_n$ can be decomposed into $\frac{m}{8}$ copies of $K_8 \square C_n$ and $n\binom{\frac{m}{8}}{2}$ copies of $K_{8,8}$. By Theorem 1.2, $K_{8,8}$ is fork-decomposable. Hence $K_m \square C_n$ is fork-decomposable.

Case 2(b). $n \equiv 1(mod\ 2)$ and $m \equiv -1(mod\ 8)$.
When $m = 7$, the graph $K_7 \square C_n$ can be decomposed into n copies of $K_7 \circ K_1$. By Theorem 1.5, $K_7 \circ K_1$ is fork-decomposable. Hence the graph $K_7 \square C_n$ is fork-decomposable. The graph $K_m \square C_n$ can be decomposed into $K_7 \square C_n$, $K_{m-7} \square C_n$ and n copies of $K_{7,m-7}$. By Theorem 1.2, $K_{7,m-7}$ is fork-decomposable and by Case 2(a), $K_{m-7} \square C_n$ is fork-decomposable. Hence $K_m \square C_n$ is fork-decomposable.

Case 3. $n \equiv 0(mod\ 4)$.
It is enough to prove the result for $m \equiv 1(mod\ 4)$ and $m \equiv 2(mod\ 4)$. Suppose $m \equiv 1(mod\ 4)$. When $m = 5$, the graph $K_5 \square C_n$ can be decomposed into $K_5 \square P_{n-1}$ and $K_5 \circ \overline{K_2}$. By Theorem 5.1, the graph $K_5 \square P_{n-1}$ is fork-decomposable and by Theorem 1.5, $K_5 \circ \overline{K_2}$ is fork-decomposable. Hence the graph $K_5 \square C_n$ is fork-decomposable. The graph $K_m \square C_n$ can be decomposed into $K_5 \square C_n$, $K_{m-5} \square C_n$ and n copies of $K_{5,m-5}$. By Theorem 1.2, $K_{5,m-5}$ is fork-decomposable. Hence $K_m \square C_n$ is fork-decomposable for $m \equiv 1(mod\ 4)$.

Suppose $m \equiv 2(mod\ 4)$. When $m = 2$, the graph $K_2 \square C_n$ is fork-decomposable by Lemma 3.1. The graph $K_m \square C_n$ can be decomposed into $K_2 \square C_n$, $K_{m-2} \square C_n$ and n copies of $K_{2,m-2}$. By Theorem 1.2, $K_{2,m-2}$ is fork-decomposable. Hence $K_m \square C_n$ is fork-decomposable for $m \equiv 2(mod\ 4)$.

Also, by using the case (1), $K_m \square C_n$ is fork-decomposable for all $n \equiv 0(mod\ 4)$.

Conclusion

In this paper we have investigated the existence of fork-decomposition of Cartesian product of graphs. Also, we have investigated the necessary and sufficient conditions for the decomposition of Cartesian product of graphs into forks. A study on the fork-decomposition of other product graphs and total graphs is finalized and will appear as a separate paper in a reputed journal.

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References

1. Abolape D. Akwu, Deborah O.A. Ajayi, Sunlet Decomposition of Certain Equipartite Graphs, Int. J. Comb., (2013), Article ID 907249, 4 pages.
2. Alspach, B., Gavlas, H., Cycle decomposition of K_n and K_n-I, J .Combin. Theory (B), 81 (2001), 77-99.
3. Barat, J., Gerbner, D., Edge decomposition of graph into copies of tree with four edges, Electron. J. Combin., 21 (1) (2014).
4. Bondy, J.A., Murty, U.S.R. , Graph Theory, Springer International Edition, 2008.
5. Caterina De Simone and Antonio Sassano, Stability number of bull and chair-free graphs, Discrete Appl. Math., 41 (1993), 121-129.
6. Chithra Devi P., Paulraj Joseph J., P_4 Decomposition of Product graphs, JP Journal of Mathematical Sciences, 7 (1 & 2) (2013), 13-39.
7. Dor, D., Tarsi, M., Graph-decomposition is NP-complete: a complete proof of Holyr conjecture, SIAM J. Comput., 26 (1997), dor, 11661187.
8. Ming-qing Zhai, Chang-hong Lu, Path Decomposition of Graph with given path length, Acta Math. Appl. Sin. Engl. Ser., 22 (4) (2006), 633-638.
9. Paulraj Joseph, J., Samuel Issacraj, A., Fork-decomposition of graphs, PreConference Proceedings of the International Conference on Discrete Mathematics, ISBN:978-93-91077-53-2, (2022), 426-431.
10. Richard Hammack, Wilfied Imrich, Samndi Klavzar, Handbook of Product Graphs, Second Edition, Taylor and Francis Group, 2011.
11. Sunil Kumar, C., On P_4-decomposition of Graphs, Taiwanese J. Math., Vol. 7 (4) (2003), pp. 657-664.
12. Yang Zhao, Baoyindureng Wu, Star decomposition of graphs, Discrete Math. Algorithms Appl., 7(2), (2015), 1550016 (9 pages).