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Sigma Chromatic Number of Some Graphs

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ABSTRACT

The Sigma colouring ($\sigma -$ colouring) of a graph G with n vertices is an injection from $V(G)$ to $\{1,2,\dots,n\}$ such that the colour sums (adding the colours of the neighbouring vertices) of any two neighbouring vertices are different. The smallest number of colours needed to colour a graph G is represented by its Sigma Chromatic number, $\sigma(G)$. In this article we obtain the σ-colouring of some graphs such as Barbell Graph, Twig graph, Shell graph, Tadpole, Lollipop, Fusing all the vertices of cycle and duplication of every edge by a vertex in C_n.

Keywords: σ- colouring, Sigma Chromatic number, Barbell Graph, Twig graph, Shell graph, Tadpole, Lollipop.

AMS Subject Classification Number: 05C15

Introduction

By a graph, we mean simple graph. Several types of graph colouring were investigated in [1,3] and new variations of colouring are available in [3,5,6]. The $\sigma -$ colouring was introduced by Gary Chartrand et.al.[1] in 2008 as he was doing a project. Gary Chartrand et.al. presented the first paper [2] in 2010, finding $\sigma(G)$ of complete graphs, complete r −partite graph with $r \geq 2$ and cycles. He proved that $\sigma(G) \leq \mathcal{X}(G)$ where $\mathcal{X}(G)$ is the minimum number of colours used in the proper vertex colouring G. Finding the Sigma Chromatic number of a graph G is the goal of the Sigma colouring problem. We begin by recollecting some basic definitions used in this article.

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Definition.1.1. Barbell graph, B_n, is obtained by connecting two copies of K_n by a bridge.

Definition 1.2. A path with two pendent edges attaching to each internal vertex forms a Twig graph.

Definition. 1.3. A Shell graph is defined as a cycle C_n with $(n-3)$ chords sharing a common end point called the apex.

Definition 1.4. The tadpole graph, $T_{n\,l}$, is the graph obtained by joining a cycle to a path P_l of lenth l.

Definition 1.5. The lollipop graph denoted by $L_{n\,l}$, is the graph obtained by joining a complete graph K_n to a path of lenth l.

Definition 1.6. Fusion (Identification) of two distinct vertices u, v of a graph G produces a new graph G_1 constructed by replacing the vertices u, v by a single vertex w such that every edge which is incident with either u or v in G is now incident with w in G_1.

Definition.1.7. The floor function of a real number x is the largest integer less than or equal to x and it is denoted by $\lfloor x \rfloor$.The ceil function of a real number x is the smallest integer greater than or equal to x and is denoted by $\lceil x \rceil$.

Definition.1.8.[3]. Imagine a vertex colouring of G which is not-proper. The function $c: V(G) \rightarrow N$ is a vertex colouring of a graph G, and $c(v)$ denote the colour of a vertex v. We encode the colours by natural numbers in order to do this.
For any $v \in V(G)$, the sum of colours of the vertices neighbouring to v be denoted by $\sigma(v)$; if for any two adjacent vertices $u, v \in V(G), \sigma(v) \neq \sigma(u)$, then the colouring is called a Sigma colouring ($\sigma -$ colouring) of G. The minimum number of colours used in a sigma colouring of G is called the sigma chromatic number of G and is denoted by $\sigma(G)$.

In our article, we obtain the σ -colouring of some graphs such as Barbell Graph, Twig graph, Shell graph, Tadpole, Lollipop, Fusing all the vertices of cycle and duplication of every edge by a vertex in C_n. For the expressions and definitions not explained in this article, we may refer to Harary[4].

Findings

Theorem.2.1. A Barbell graph, B_n is $\sigma -$ colourable and $\sigma(B_n) \leq n + 1$.

Proof: Consider B_n, the barbell graph constructed by connecting two copies of complete graph K_n and K_n' by a bridge. Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices of K_n and $v_1', v_2', v_3', \ldots, v_n'$ be the vertices of K_n' and the bridge be $e = v_1 v_1'$.

Define c : V $(B_n) \rightarrow \{1,2\}$ as follows:

$c(v_i) = i$; if $1 \leq i \leq n$.

$c(v_i') = i + 1$; if $1 \leq i \leq n$.

The vertices v_i and v_{i+1}, $(2 \leq i \leq n - 1)$ are of the same degree and are adjacent in K_n. We colour all the vertices with different colours, otherwise these vertices will receive the same colour sum, which violates the rules of $\sigma -$ colouring. In the case of K_n', if we colour all the vertices with same set of colours in K_n, then atleast two vertices receives the same colour sum, which breaks the condition of $\sigma -$ colouring. So, we use an additional colour $n + 1$ in K_n'. Then, all adjacent vertices get different vertex sum. Here c is a $\sigma -$ colouring with $\sigma(G) \leq n+1$.

Theorem.2.2. For any Twig graph T_m, $m \geq 2$, $\sigma(T_m) = 2$.

Proof: Let the initial and terminal vertices of the path be v_1 and v_{m+2} and let $v_2, v_3, \ldots, v_{m+1}$ be the internal vertices of the path. Let v_i $(1 \leq i \leq m + 2)$, $u_{j,}$ $(1 \leq j \leq m)$ and $w_{j,}$ $(1 \leq j \leq m)$ be the vertex set and $v_i v_{i+1}$ $(1 \leq i \leq m + 1)$, $u_j v_{j+1} (1 \leq j \leq m)$, $w_j v_{j+1} (1 \leq j \leq m)$ be the edge set.

Define c : V$(T_n) \rightarrow \{1,2\}$ as follows.

$c(v_i)$= 1 if i is odd.

$c(v_i)$= 2 if i is even.

$c(u_{i,}) = 1$

$c(w_{i,}) = 1$

It could be noted all the adjacent vertices get different vertex sum. Here c is a $\sigma -$ colouring with $\sigma(T_m) \leq 2$. If possible, consider

$\sigma(T_m) = 1$. Since the vertices v_i and v_{i+1} ($2 \leq i \leq m$) are of the same degree and we colour all the vertices with the same colour 1 these adjacent vertices v_i and v_{i+1}($2 \leq i \leq m$) get the same colour sum, which contradicts the rule of $\sigma -$ colouring. So $\sigma(T_m) \neq 1$. Hence, $\sigma(T_m) = 2$.

Theorem.2.3. The Shell graph is $\sigma -$ colourable and $\sigma(S_{n,n-3}), n \geq 4, = 2$.

Proof: Let $v_1, v_2, v_3, \ldots, v_n$ be the nodes of the cycle C_n. Let v_1 be the apex vertex of shell graph $S_{n,n-3}$.

Suppose the case where the number of nodes is odd.

Define $c : V(S_{n,n-3}) \rightarrow \{1,2\}$ as follows:

$c(v_{2i}) = 2; 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$.

$c(v_{2i-1}) = 1; 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$.

$c(v_n) = 2$.

Suppose the case where the number of nodes is even.

Define the vertex colour $c : V(S_{n,n-3}) \rightarrow \{1,2\}$ as follows:

$c(v_{2i}) = 2; 1 \leq i \leq \frac{n}{2}$.

$c(v_{2i-1}) = 1; 1 \leq i \leq \frac{n}{2}$

It could be noted all the adjacent vertices get different vertex sum. Here c is a $\sigma -$ colouring with $\sigma(S_{n,n-3}) \leq 2$. If possible, consider $\sigma(S_{n,n-3}) = 1$. Since the vertices v_i and v_{i+1} ($3 \leq i \leq n-2$) are of the same degree and we colour all the vertices with the same colour 1, these adjacent vertices v_i and v_{i+1} ($3 \leq i \leq n-2$) get the same colour sum which contradicts the rule of $\sigma -$ colouring. So, $\sigma(S_{n,n-3}) \neq 1$. Hence, $\sigma(S_{n,n-3}) = 2$.

Theorem.2.4. The Tadpole graph, $T_{n\,l}$, is $\sigma -$ colourable. For $n \geq 3$, $l \geq 3$, $\sigma(T_{n\,l}) = 2$.

Proof: Let $T_{n\,l}$ be the graph obtained by joining a cycle C_n by a path P_l. Let the cycle C_n have vertices $v_1, v_2, v_3, \ldots, v_n$ and $u_1, u_2, u_3, \ldots, u_l$ be the vertices of the path joining v_1 *to* P_l. Define $c : V(T_{n\,l}) \rightarrow \{1,2\}$ as follows.

Case I: When n is odd.

$c(v_{2i}) = 2; 1 \leq i \leq \frac{n-1}{2}$.
$c(v_{2i-1}) = 1; 1 \leq i \leq \frac{n+1}{2}$.
$c(u_{4i-2}) = 1,\ 1 \leq i \leq \left\lfloor \frac{n+2}{4} \right\rfloor$.
All other vertices in the path except u_{4i-2} ($1 \leq i \leq \left\lfloor \frac{n+2}{4} \right\rfloor$) are coloured with colour 2.
Case II: When n is even.

$c(v_{2i}) = 1; 1 \leq i \leq \frac{n}{2}$.

$c(v_{2i-1}) = 2; 1 \leq i \leq \frac{n}{2}$.

$c(u_{4i-3}) = 1,\ 1 \leq i \leq \left\lfloor \frac{n+3}{4} \right\rfloor$. All other vertices in the path except $u_{4i-3}(1 \leq i \leq \left\lfloor \frac{n+3}{4} \right\rfloor)$ are coloured with colour 2.

It could be noted all the adjacent vertices get different vertex sum. Here c is a $\sigma -$ colouring with $\sigma(T_{n\,l}) \leq 2$. If possible, consider $\sigma(T_{n\,l}) = 1$. Since the vertices v_i and $v_{i+1}(2 \leq i \leq n-1)$ are of the same degree and we colour all the vertices with the same colour 1 these adjacent vertices v_i and $v_{i+1}(2 \leq i \leq n-1)$ get the same colour sum ,which violates the rule of $\sigma -$ colouring. So, $\sigma(T_{n\,l}) \neq 1$. Hence, $\sigma(T_{n\,l}) = 2$.

Theorem.2.5. The Lollipop graph $L_{n\,l}$ is $\sigma -$ colourable. For $n \geq 3$, $l \geq 3$, $\sigma(L_{n\,l}) \leq n$.

Proof: Let $L_{n\,l}$ be the graph obtained by joining a complete graph K_n by a path P_l.

Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices of a complete graph K_n and $u_1, u_2, u_3, \ldots, u_l$ be the vertices of the path joining v_1 to P_l.

Define the vertex colour $c : V(L_{n\,l}) \rightarrow \{1,2,3,\ldots,n\}$ as follows.

$c(v_i) = i; 1 \leq i \leq n$.

$c(u_{4i-2}) = 1,\ 1 \leq i \leq \left\lfloor \frac{l+2}{4} \right\rfloor$. All other vertices in the path except $u_{4i-2}(1 \leq i \leq \left\lfloor \frac{l+2}{4} \right\rfloor)$ in the path are coloured with colour 2.

The vertices v_i and $v_{i+1}(2 \leq i \leq n-1)$ are of the same degree and are adjacent in K_n. We colour all the vertices with different colours, otherwise these vertices get the same colour sum, which contradicts the rule of $\sigma -$ colouring. Using the above colouring pattern all adjacent vertices get different vertex sum. Here, c is a $\sigma -$ colouring with $\sigma(L_{n\,l}) \leq n$.

Theorem.2.6. The graph obtained by joining two copies of C_n by a path P_m admits $\sigma -$ colouring. For $n \geq 3, m \geq 3$ $\sigma(H) = 2$.

Proof: Consider H be the graph constructed by joining two copies of C_n by a path P_m.

Let $v_1, v_2, v_3, \ldots, v_n$ be the nodes of the first copy of C_n, $u_1, u_2, u_3, \ldots, u_n$ be the nodes of the other copy of C_n and $w_1, w_2, w_3, \ldots, w_m$ be the nodes of path P_m.

Suppose the case where the number of nodes of $\boldsymbol{C_n}$ is odd.

Case I: When n is odd.

(a) When m is odd.

Define $c : V(H) \rightarrow \{1,2\}$ as follows

$$c(w_i) = c(u_i) = c(v_i) = \begin{cases} 2 \; if \; i \; is \; even \\ 1, if \; i \; is \; odd \end{cases}$$

(b) When m is even.

$$c(v_i) = \begin{cases} 2 \; if \; i \; is \; even \\ 1, if \; i \; is \; odd \end{cases}$$

$$c(u_i) = \begin{cases} 2 \; if \; i \; is \; even \\ 1, if \; i \; is \; odd \end{cases}$$

$$c(w_i) = \begin{cases} 2 \; if \; i \; is \; even, i \neq m \\ 1, if \; i \; is \; odd \end{cases}$$

$$c(w_{m,}) = 1$$

Case II: When n is even.

(a): When m is even.

Define $c : V(H) \rightarrow \{1,2\}$ as follows

$$c(w_i) = c(u_i) = c(v_i) = \begin{cases} 2 \; if \; i \; is \; even \\ 1, if \; i \; is \; odd \end{cases}$$

(b): When m is odd.

$$c(v_i) = \begin{cases} 2\ if\ i\ is\ even \\ 1\ ,if\ i\ is\ \ odd \end{cases}$$

$$c(u_i) = \begin{cases} 2\ if\ i\ is\ even \\ 1\ ,if\ i\ is\ \ odd \end{cases}$$

$$c(w_i) = \begin{cases} 2\ if\ i\ is\ even, i \neq m \\ 1\ ,if\ i\ is\ \ odd \end{cases}$$

$$c(w_m) = 1$$

It could be noted all the adjacent vertices get different vertex sum. Here c is a $\sigma -$ colouring with $\sigma(H) \leq 2$. If possible, consider $\sigma(H) = 1$. Since the vertices v_i and $v_{i+1}(2\leq i \leq n-2)$ are of the same degree and we colour all the vertices with the same colour 1. These adjacent vertices v_i and $v_{i+1}(2\leq i \leq n-2)$ get the same colour sum, which contradicts the rule of $\sigma -$ colouring. So, $\sigma(H) \neq 1$. Hence $\sigma(H) = 2$.

Theorem.2.7. The graph obtained by fusing all the n vertices of cycle C_n with the apex vertices of n copies of $K_{1\,m}$ admits $\sigma -$ colouring. $For\ \ n > 3, m \geq 3, \sigma(H) = 2$.

Proof: Consider H be the graph obtained by by fusing all the n vertices of cycle C_n with the apex vertices of n copies of $K_{1\,m}$. Let $v_1, v_2, v_3,\ldots, v_n$ be the vertices of the cycle C_n. Fusing all the vertices $v_i$of cycle C_n with the apex vertices of star $K_{1\,m}$by $v_{i\,j}$, $1\leq i \leq n$, $1\leq j \leq m$.

Suppose the case where the number of nodes of $\boldsymbol{C_n}$ is odd.

Define $c : V(H) \rightarrow \{1,2\}$ as follows.

$$c(v_i) = \begin{cases} 2\ if\ i\ is\ even \\ 1\ ,if\ i\ is\ \ odd \end{cases}$$

$c(v_{i\,j}) = 1; 1\leq i \leq n-1, 1\leq j \leq m-1.$

$$c(v_{n\,m}) = 2.$$

Suppose the case where the number of nodes of $\boldsymbol{C_n}$ is even.

Define $c\ : V(H) \rightarrow \{1,2\}$ as follows.

$$c(v_i) = \begin{cases} 2\ if\ i\ is\ even \\ 1\ ,if\ i\ is\ \ odd \end{cases}$$

$c(v_{i\,j}) = 1; 1\leq i \leq n, 1\leq j \leq m.$

It could be noted all the adjacent vertices get different vertex sums. Here c is a $\sigma -$ colouring with $\sigma(G) \leq 2$. If possible, consider $\sigma(G) = 1$. Since the vertices v_i and $v_{i+1}(1\leq i \leq n-1)$ are of the same degree and we colour all the vertices with the same colour 1, these adjacent vertices v_i and $v_{i+1}(1\leq i \leq n-1)$ get the same colour sum, which contradicts the rule of $\sigma -$ colouring. So, $\sigma(G) \neq 1$. Hence, $\sigma(H) = 2$.

Theorem.2.8. The graph constructed by replication of every edge replaced with a vertex in C_n is $\sigma -$ colouring , $n \geq 3$, $\sigma\ (H) = 2$.

Proof:Consider $v_1, v_2, v_3,\ldots, v_n$ be nodes of the cycle C_n. Let H be the graph constructed by replication of every edge $v_1v_2\ , v_2v_3, \ldots, v_n\ v_1$ in C_n by the corresponding new nodes $u_1, u_2, u_3,\ldots, u_n$,respectively.

Suppose the case where the number of nodes of C_n is odd.

Define $c\ : V(H) \rightarrow \{1,2\}$ as follows

$c(v_{2i}) = 2; 1\leq i \leq \frac{n-1}{2}$.

$c(v_{2i-1}) = 1; 1\leq i \leq \frac{n+1}{2}$

$c(u_i) = 1; 2\leq i \leq n.$

$c(u_1) = 2.$

Suppose the case where the number of nodes of $\boldsymbol{C_n}$ is even.

Define $c\ : V(H) \rightarrow \{1,2\}$ as follows

$c(v_{2i}) = 2; 1\leq i \leq \frac{n}{2}$.

$c(v_{2i-1}) = 1; 1\leq i \leq \frac{n}{2}$

$c(u_i) = 1; 1 \leq i \leq n.$

It could be noted all the adjacent vertices get different vertex sum. Here, c is a $\sigma -$ colouring with $\sigma(G) \leq 2$. If possible, consider $\sigma(G) = 1$ .Since the vertices v_i and $v_{i+1}(1\leq i \leq n-1)$ are of the same degree and we colour all the vertices with the same colour 1, these adjacent vertices v_i and $v_{i+1}(1\leq i \leq n-1)$ get the same colour sum, which contradicts the rule of $\sigma -$ colouring. So, $\sigma(G) \neq 1$. Hence $\sigma(G) = 2$.

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