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Double Geodetic Number of a Line Graph

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Abstract

Any line graph $L(G)$, the vertices correspond to the edges of $G(V, E)$ and two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent". "If there are vertices u, v in S such that $x, y \in I[u, v]$ for any pair of vertices x, y in G, then the set S of vertices of G is said to be a double geodetic set of G. The lowest cardinality of a double geodetic set is represented by the double geodetic number $dg(G)$". In this study, we determine double geodetic number of several line graphs.

Keywords: double geodetic number, line graph, cartesian product, vertex covering number.

Introduction

A connected finite undirected graph with no loops or multiple edges is referred to as a graph, $G = (V, E)$. The standard notation for the number of edges and vertices in a graph G is $m = |E|$ and $n = |V|$. We cite [3]. If the subgraph induced by a vertex's neighbours is complete, then that vertex is an extreme vertex of G. The closed interval $I[x, y]$ consists of all vertices lying on some x-y geodesic of G, while for $S \subseteq V$, $I[S] = \cup_{x,y \in S} I[x, y]$. A set of vertices S is said to be a geodetic set if $I[S] = V$ and the geodetic number is the lowest cardinality of a geodetic set which is denoted by $g(G)$.In [1] and [2],

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the geodetic number is presented and briefly discussed. The double geodetic number that [4] first introduced.

Basic Results

The following theorem is needed for this paper's results to be supported.

Theorem 2.1 [4] For the cycle C_n of order $n \geq 3$, $dg(C_n) = \begin{cases} 2, & if \ n \ is \ even \\ n, & if \ n \ is \ odd \end{cases}$.

Double geodetic number of a line graph

Definition 3.1. A set S' of vertices of $L(G) = H$ is said to be double geodetic set of H if for each pair of vertices x, y in H there exist vertices u, v in S' such that $x, y \in I[u, v]$. The double geodetic number is the lowest cardinality of the double geodetic set of $L(G)$ and is denoted by $dg[L(G)]$.

Example 3.2

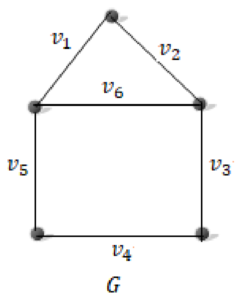

Figure 3.1

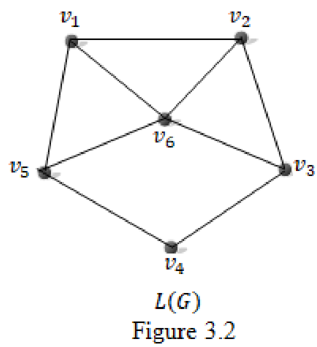

Figure 3.2

In Figure 3.2, $L(G)$ is the line graph of In $L(G), S_1 = \{v_1, v_3, v_5\}$ is the minimum geodetic set but S_1 is not a double geodetic set of $L(G)$ and neither $3-$ element nor $4-$ element subset of vertices of $L(G)$ contains the dg-set of $L(G)$. Also, it is obvious that, the set $S_1 = \{v_1, v_2, v_3, v_4, v_5\}$ is the minimum double geodetic set of $L(G)$. Therefore, $g[L(G)] = 3$ and $dg[L(G)] = 5$. Consequently, a line graph's geodetic number and double geodetic number may differ.

Theorem 3.3 For the line graph $L(G)$ of order n, Then $2 \leq g[L(G)] \leq dg[L(G)] \leq n$.

Proof. A geodetic set requires two vertices at a minimum. Therefore $g[L(G)] \geq 2$. We know that, each geodetic set must contain a double geodetic set. Then $g[L(G)] \leq dg[L(G)]$. Since all the vertices of $L(G)$, is a double geodetic set of $L(G)$, $dg[L(G)] \leq n$.

Theorem 3.4 For any line graph $L(G)$ of order n, $g[L(G)] = 2$ iff $dg[L(G)] = 2$.

Proof. Firstly, we assume that $dg[L(G)] = 2$. We prove that $g[L(G)] = 2$. Since $dg[L(G)] = 2$. By using Theorem 3.3, we get $g[L(G)] = 2$. Conversely, we assume that $g[L(G)] = 2$. To prove that $dg[L(G)] = 2$, suppose we assume that $dg[L(G)] \neq 2$. We know that $G$is connected. By Property 3.2.1 in [7], $L(G)$ is connected. It follows from Proposition 2.14 in [4], $dg[L(G)]$. This conflicts with our assumption. Hence, $dg[L(G)] = 2$.

Theorem 3.5 For every tree T with k end edges, $dg[L(T)] = k$.

Proof. Let S be the collection of each extreme vertices of the line graph $L(T)$. By Theorem 2.5 in [4], $dg[L(T)] \geq |S|$. Further more, each double geodetic set of T contains every extreme vertex of a line graph $L(T)$. The extreme vertices of $L(T)$ are the corresponding end edges of T. So $dg[L(T)] \leq |S|$. By Corollary 2.9 in [4] , $dg[L(T)] = |S| = k$. Hence, $dg[L(T)] = k$.

corollary 3.6 For any path P_n with n vertices , $dg(L(P_n)] = 2$.

Proof. It is clear that $g[L(P_n)] = 2$. By Theorem 3.4, $dg[L(P_n)] = 2$.

Theorem 3.7 For a nontrivial tree T of order n and d be the diameter, then $dg[L(T)] \leq n - d + 1$.

Proof. Let T be any nontrival tree of order n and d be the diameter. Let q be the vertices of $L(T)$. Let $p = v_0, v_1, v_2, \dots, v_d = q$ be a path for which $d(p,q) = d$. Let S be the extreme vertices of $L(T)$also let $S = V[L(T)] - \{v_1, v_2, \dots, v_{d-1}\}$. Neccesarily, by Theorem 3.5, $dg[L(T)] = k \leq |S| = n - (d-1) = n - d + 1$.

Theorem 3.8 For cycle C_n of order $n \geq 3$, $dg[L(C_n)] = \begin{cases} 2, & if\ n\ is\ even \\ n, & if\ n\ is\ odd \end{cases}$.

Proof. This statement is true based on Theorem 2.1

Theorem 3.9 For the helm graph H_n , $dg[L(H_n)] = \begin{cases} 8, & if \ n = 4 \\ 3n, & if \ n \geq 5 \end{cases}$.

Proof. Let x the vertex of K_1 , $V(C_n) = \{ v_1, v_2, v_3, \dots, v_n \}$, $E = \{e_1, e_2, \dots, e_n\}$ be the internal edges and $U = \{u_1, u_2, \dots, u_n\}$ be the degree one vertices in helm graph H_n . Now, the vertices $W = \{ w_1, w_2, \dots, w_n \}$, are formed from the end edges of H_n; $W \subseteq V[L(H_n)]$, and $X = \{x_1, x_2, , \dots, x_n\}$ are the vertices made up of the edges of C_n; $X \subseteq V[L(H_n)]$, $Y = \{y_1, y_2, \dots, y_n\}$ which are the vertices of $L(H_n)$, made up of internal edges of H_n; $Y \subseteq V[L(H_n)]$.

Case (i) If $n = 4$.

For the graph $L(H_4)$, the set of vertices in the set $W = \{w_1, w_2, w_3, w_4\}$ are all extreme vertices. The set W is the only minimum geodetic set of $L(H_4)$, but this set W is not double geodetic set. Because, some pair of vertices (w_i, y_i) where $1 \leq i \leq 4$, does not lie on any geodesic of W. Now, consider the set $Y = \{y_1, y_2, y_3, y_4\}$. All are weak extreme vertices. Hence, the set $W \cup Y$ is unique minimum double geodetic set in $L(H_4)$. Thus, we get $|W \cup Y| = 8$. Therefore, $dg[L(H_4)] = 8$.

Case (ii) For $n \geq 5$.

Let v be any vertex in $L(H_n)$. First we prove that v is $L(H_n)$'s weak extreme vertex. Let v' be the eccentric vertex of v in $L(H_n)$. Then, v, v' lie only on $I[v, v']$ so that $L(H_n)$ has a weak extreme vertex v. Proceeding like this, all vertices of $L(H_n)$ are weak extreme vertices. By Proposition 2.14 in [4], All the vertices of H_n are unique double geodetic set of $L(H_n)$ and $|W \cup X \cup Y| = 3n$, Thus, $dg[L(H_n)] = 3n$.

Corollary 3.10: For the helm graph H_n , $n \geq 5$, $g[L(H_n)] + dg[L(H_n)] = m + n$.

Proof. helm graph H_n has $3n$ edges. It becomes $3n$ vertices in $L(H_n)$. Since $g[L(H_n)] = n$ and $dg[L(H_n)] = 3n$ and $V[L(H_n)] = E(H_n) = m$ and $V(W) = n$,where W is the extreme vertices of $L(H_n)$.

Now, $g[L(H_n)] + dg[L(H_n)] = 4n = 3n + n = V[L(H_n)] + V(W) = m + n$.

Corollary 3.11: For the helm graph $(n \geq 5)$, $dg[L(H_n)] = \delta\Delta - 6$.

Proof. $L(H_n)$ has a minimum degree δ of 3 and a maximum degree Δ of $n+2$.

Now, $dg[L(H_n)] = 3n$, $dg[L(H_n)] + 6 = 3n + 6 = 3(n+2) = \delta\Delta$. $dg[L(H_n)] = \delta\Delta - 6$.

Theorem 3.12 For the wheel graph of order $n \geq 7, dg[L(W_n)] = n - 1$.

Proof. Let $W_n = K_1 + C_{n-1}$ $(n \geq 7)$ with x as the vertex of K_1 and $(C_{n-1}) = \{v_1, v_2, \dots, v_{n-1}\}$, $E = \{e_1, e_2, \dots, e_{n-1}\}$ be the internal edges of W_n. Now, $Y = \{y_1, y_2, \dots, y_{n-1}\}$ be the vertices made up of the edges of C_{n-1} . i.e) $Y \subseteq V[L(W_n)], Z = \{z_1, z_2, \dots, z_{n-1}\}$ which verticesof $[L(W_n)]$ formed from the internal edges of W_n; $Z \subseteq V[L(W_n)]$. For every pair of vertices which are $d(u, v) = diam[L(W_n)]$ is formed by the double geodetic set of $L[(W_n)]$. Obviously, the collection of all vertices of the set Y is a $dg -$ set of $L(W_n)$ and $dg[L(W_n)] = n - 1$.

Theorem 3.13 For the friendship graph F_n having $2n + 1$ vertices, $dg[L(F_n)] = n$- $n \geq 3$.

Proof. friendship graph F_n has $2n + 1$ vertices and $3n$ edges. Let x be common vertex. $2n$ edges are incident with common vertex x. This $2n$ edges forms $2n$ vertices $U = \{u_1, u_2, \dots, u_{2n}\}$ in (F_n) . Also the remaining n edges of F_n which are not incident with the vertex x forms n extreme vertices $W = \{w_1, w_2, \dots, w_n\}$ in $L(F_n)$; $U, W \subseteq V[L(F_n)]$. By Theorem 2.5 in [4], the set S contains the vertices of W and $d(u, v) = diam[L(F_n)]$ and every pair of vertices lies on the set S. Thus, S is the only minimum double geodetic set of $L(F_n)$and so $|S| = n$.

Corollary 3.14 For the friendship graph F_n, $(n \geq 3)$, $g[L(F_n)] + dg[L(F_n)] = m - n$.

Proof. Let $U = \{u_1, u_2, \dots, u_{2n}\}$ be the vertices made of the internal edges of F_n and $W = \{w_1, w_2, \dots, w_n\}$ be the extreme vertices of $L(F_n)$ formed from n −copies of the cycle graph C_3 of F_n. W forms the minimum double geodetic set of $L(F_n)$. It is obvious that $g[L(F_n)]$ and $dg[L(F_n)]$ are same. Since the friendship graph F_n has $2n$ internal edges, it becomes $2n$ vertices of $L(F_n)$. Since $V[L(F_n)] = E(F_n)] = m$ and $V(W) = n$ and also $g[L(F_n)] = n$.

Now, $g[L(F_n)] + dg[L(F_n)] = 2n = V(U)$

$$= V[L(F_n)] - V(W)$$

$$= m - n.$$

Corollary 3.15 For the friendship graph F_n, $(n \geq 3)$, $dg[L(F_n)] = \frac{\Delta}{\delta}$.

Proof. Minimum degree (δ) of $L(F_n)$ is 2 and maximum degree (Δ) of $L(F_n)$ is $2n$.

Now, $dg[L(F_n)] = n$

$$=\frac{2n}{2}$$

$$=\frac{\Delta}{\delta}.$$

Theorem 4.1 For the pan graph P_n of order $n \geq 3$, $dg[L(P_n)] = \begin{cases} 2 & if\ n\ is\ odd \\ 4 & if\ n\ is\ even \end{cases}$.

Proof. Consider a cycle $\{v_1, v_2, \dots, v_n, v_1\}$ with n vertices. Let P_n be the pan graph made from $G = C_n$ by adding an end edge uv such that $u \in G$ and $v \notin G$, by the definition of line graph, cycle's line graph is also a cycle and the end edge in $P_n{}'$ is the extreme vertex of $L(P_n)$. Now, $L(P_n) = C_n \cup K_3$. We prove the following cases.

Case (i) n is odd

The geodetic number of $L(P_n)$ is 2. By theorem 3.4, $dg[L(P_n)\ ] = 2$.

Case (ii) n is even

Since the edge $uv = v_k$ the extreme vertex in $L(P_n)$. By theorem 2.5 in [4] , v_k belongs to the double geodetic set of $L(P_n)$. Since $L(P_n) = C_n \cup K_3$–the edges v_i, v_j occurring on the vertex of u , which is antipodal in P_n- are the vertices in $L(P_n)$. These vertices are contained in the double geodetic set of $L(P_n)$. Let v_m be the vertex of $L(P_n)$ which is the eccentric vertex of v_k . This follows from Case (ii) of theorem 3.9, v_m is a weak extreme vertex of $L(P_n)$. By Proposition 2.14 in [4], v_m belongs to the double geodetic set. Hence, $S = \{v_k, v_i, v_j, v_m\}$ is the double geodetic set of $L(P_n) and\ hence\ dg[L(P_n)] = 4$.

Theorem 4.2 For the pan graph P_n , n is odd, $dg[L(P_n)] = 2 \propto_\circ (P_n) - n + 1$.

Proof. If $n \geq 3$ is odd and let $\propto_\circ$ be the vertex covering number of P_n. *Since* $dg[L(P_n)] = 2$ and n is odd, $\propto_\circ (P_n) = \frac{n+1}{2}$. Hence, $dg[L(P_n)] = 2 = 1 + 1 = 1 - n + 1 + n$ and $dg[L(P_n)] = \frac{2(-n+1+1+n)}{2} = \frac{2(1-n)}{2} + \frac{2(1+n)}{2} = 2 \propto_\circ (P_n) - n + 1$.

Theorem 4.3 For the pan graph P_n , n is even, $dg[L(P_n)] = 2 \propto_\circ (P_n) - n + 2$.

Proof. Let $\propto_\circ$ is the vertex covering number of P_n ,$n \geq 3$, n is even. We have $dg[L(P_n)] = 4$ and n is even, $\propto_\circ (P_n) = \frac{n+2}{2}$. Hence, $dg[L(P_n)] = 4 = 2 + 2 = -n + 2 + n + 2$.

$$= \frac{2(-n + 2 + n + 2)}{2} = 2 \propto_\circ (P_n) - n + 2.$$

Theorem 4.4 If the graph G' is obtained by adding an end edge u_i,v_i , $i = 1,2, \dots, n$ to each vertex of $G = C_n$ such that $u_i \in G$, $v_i \notin G$ .Then, $dg[L(G')] = \begin{cases} 2n, for\ n\ is\ odd \\ n,\ \ for\ n\ is\ even \end{cases}$.

5. Cartesian Product

Theorem 5.1. For any cycle C_n of order $n \geq 3$, $dg[L(C_n \times P_2)] = \begin{cases} 4 & if\ n\ is\ even \\ 3n & if\ n\ is\ odd \end{cases}$.

Proof. Let $C_n \times P_2$ be formed from two copies G_1 and G_2 of C_n. this graph is called $n -$ prism graph. The $C_n \times P_2$ graph contains two sets of cycle C_n. One set of cycle is C_1 and another one is C_2. In $L(C_n \times P_2)$, the vertices $X = \{x_1, x_2, \dots, x_n\}$ corresponds the edges of $C_1$and the edges of C_2 converted to the vertices $Y = \{y_1, y_2, \dots, y_n\}$ .also, the set $Z = \{z_1, z_2, \dots, z_n\}$ corresponds to edges incident with the cycles C_1 and C_2.

Case (i) if n is even

In $L(C_n \times P_2)$, the vertex x_i where $(1 \leq i \leq n)$ is an eccentric vertex of vertex x_j , $(1 \leq j \leq n)$ in X. It is obvious the pair x_i,x_j of vertices lie only $I[x_i, x_j]$. consequently, the vertex x_i and x_j are the weak extreme vertices. By Proposition 2.14 in [4], the vertices x_i and x_j belongs to ' , where S' is the geodetic set. But every pair does not lie on any geodesic of S'. So, we consider the set Y, where the vertices y_i and y_j are eccentric for each other. hence, the vertices $y_i$and y_j belongs to the double geodetic set S'. thus, $S' =$

$\{x_i, x_j, y_i, y_j\}$ is the minimum double geodetic set of $L(C_n \times P_2)$ .thus, $dg[L(C_n \times P_2)] = 4$.

Case (ii) if n is odd

This follows from the case (ii) of theorem 3.9.

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