Mapana – Journal of Sciences
2023, Vol. 22, Special Issue 1, *129-138*
ISSN 0975-3303 | https://doi.org/10.12723/mjs.sp1.10

The Sextuple Complete Partitions of Integers

P. Geetha*

Abstract

This paper presents the concepts of sextuple (6 – tuple) complete partitions of integers. An attempt has been made at the theorem based on the last part of sextuple complete partitions of integers.

Keywords: Integers, Compositions, Partitions, Complete Partitions

1. Introduction

A partition ν of the integer *n* is a representation of *n* as a sum of positive integers wherein the order of the summands is considered irrelevant. Let $\nu = (\nu_{0,}, \nu_1, \ldots. \nu_n)$ be a partition of the natural number m into $n+1$ parts ν_i arranged in non-decreasing order, $m = \nu_0 + \nu_1 + \ldots. + \nu_n$, $1 \le \nu_0 \le \nu_1 \le \ldots. \le \nu_n$. The sum of the parts is called the weight of the partition and is denoted by $|\nu|$, while $n+1$ is the length of the partition [1]. Complete partitions were introduced by the author in [2]. Mac Mahon [3] introduced perfect partitions of a number. He took into the consideration the number of perfect partitions of the number $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots.. -1$, where $p_1, p_2, \ldots.$ are primes, and discovered that the quantity of perfect partitions of this number is equal to the quantity of compositions of the multipartite number $(\alpha_1, \alpha_2, \ldots.)$. The fact that the number of

*Department of Mathematics, Periyar Maniammai Institute of Science & Technology, Periyar Nagar, Vallam, Thanjavur; maxgeetha@gmail.com

perfect partitions of n is the same as the number of ordered factorizations of $n+1$ was also shown. He also demonstrated the similarity between the number of ordered factorizations of $n+1$ and the number of perfect partitions of n. More recently, the concept of representation as the sum of specified numbers was applied. The phrase "complete" appears to have initially emerged in a Hoggatt and King in [4], which was then resolved in Brown's work from [5]. A similar idea of representing as a sum of given numbers was used in later days. It seems that the word “complete” first appeared in a problem suggested by Hoggatt and King in [4] which was solved in Brown’s paper[5]. They called an arbitrary sequence $\{f_i\}_{i=1}^{\infty}$ of positive integers “complete” if every positive integer n could be represented in the form $n=\sum_{i=1}^{\infty}\alpha_i f_i,$ where each α_i was either 0 or 1. We can also write a complete partition of an integer n is a partition $\nu=(\nu_1,\nu_2,....,\nu_k)$ of n, with $\nu_1=1,$ such that each integer r, $1\le r\le n,$ can be represented as a sum of elements of $\nu_1,\nu_2,....,\nu_k$. [6],[8], [9], [10].

2. Survey of Literature

J. J. Sylvester (1857) initiated the study of the partition of numbers. In 1871, he developed the partition of an even number into two prime numbers and published his findings in a paper titled "A note on the theory of a point in partitions". In 1882, he continued his research on sub invariants, or semi-invariants, to binary quantics of an unlimited order, with rational fractions and partitions. Hansraj Gupta (1969) presented a historical survey of some aspects of the theory of partitions in his work partitions – A survey. A. K. Agarwal and M. V. Subbaro (1991) presented some properties of perfect partition function and Combinatorial interpretation of n! [12, 13 & 14]. Seung Kyung Park (1996) contributed to the study of complete partitions, recurrence relations and generating functions of complete partitions. Seung Kyung Park (1997) worked on the study of the enumeration of r – complete partitions and a generalization of complete partitions of a positive integer [15]. Enumeration of geometric configurations on a convex polygon was created by Marc Noy in 1999. The theory of partitions has a fundamental invariant, according to Alladi (1999). Hokyu Lee and Seung Kyung Park (2002) represented the double complete partitions with more specified completeness and worked for the r – subcomplete partitions [16]. Ncvillc Robbins (2002) presented the convolution–type formulas for the number of partitions of n that are not divisible by r, coprime to r in the paper on partition functions and divisor sums [17]. Overpartitions were created by Jeremy Lovejoy and Sylvie Corteel in 2003. Work on a finite set's partition function was done in 2003 by T.C. Brown et al. James A. Sellers, Andrew V. Sills and Gary L. Mullen (2004) worked on bijections and congruences for generalizations of partition identities of Euler and Guy [18]. James A. Seller (2004) published the results that deal with partition functions that exclude specific polygonal numbers as parts [19]. C. S. Srivatsan et al. (2006) contributed to gentle statistics and constrained partitions. Hokyu Lee (2006) generalized the perfect partition and found a relation with ordered factorizations [20]. Oystein J. Rodseth (2006) presented the study of enumeration of M – partitions, weak M- partitions and generating functions [21]. Mac Mahon (2006) initiated the study of double perfect partitions and found a relation with ordered factorizations. Oystein J. Rodseth (2007) produced some standard results on generating functions and completeness of minimal r – complete partitions [22]. Significant observations regarding the parity of the total number of parts in odd-part partitions were made by James A. Seller in 2007.

3. Preliminaries

Definition 3.1: A complete partition of an integer n is a partition $\nu = (\nu_1 \nu_2 \ldots . \nu_k)$ of n, with $\nu_1 = 1$, such that each integer $i, 1 \le i \le n$, can be represented as a sum of elements of $\nu_1\nu_2 \ldots\nu_k$. In other words, each i can be expressed as $\sum_{j=1}^{k}\beta_j\nu_j$, where β_j is either 0 or 1.

Definition 3.2 : A double complete partition [7] of an integer *n* is a partition $\nu = (\nu_1^{m_1}\nu_2^{m_2} \ldots\ldots \nu_l^{m_l})$ of *n* such that each integer *m*, with $2 \le m \le n-2$, *n* can be represented by at least two different ways as a sum $\sum_{i=1}^{l}\beta_i\nu_i$ with $\beta_i \in \{0, 1, 2, \ldots m_l\}$.

Definition 3.3: For any integer $n \ge 8$, its triple complete partition [11] of an integer *n* is a partition $\nu = (\nu_1^{m_1}\nu_2^{m_2} \ldots\ldots \nu_l^{m_l})$ of *n* such that each integer *m*, with $3 \le m \le n-3$, can be represented at least three different ways as a sum $\sum_{i=1}^{l}\beta_i\nu_i$ with $\beta_i \in \{0, 1, 2, \ldots m_l\}$.

Definition 3.4: For any integer $n \ge 11$, the quadruple complete partition of an integer *n* is a partition $\nu = (\nu_1^{m_1}\nu_2^{m_2} \ldots\ldots \nu_l^{m_l})$ of *n* such that each integer *m*, with $4 \le m \le n-4$, can be represented at least four different ways as a sum $\sum_{i=1}^{l}\beta_i\nu_i$ with $\beta_i \in \{0, 1, 2, \ldots m_l\}$.

Definition 3.5: For any integer $n \geq 16$, its quintuple complete partition can be obtained by taking the parts of n as $\nu = (\nu_1^{m_1} \nu_2^{m_2} ..... \nu_l^{m_l})$ of n such that each integer r, with $5 \leq r \leq n-5$, can be represented by at least five different ways as a sum $\sum_{i=1}^{l} \beta_i \nu_i$ with $\beta_i \in \{0, 1, 2, ...m_l\}$.

4. Main Results

Now, we define the sextuple (6- tuple) complete partitions of integers. For both quintuple and sextuple complete partitions of integers n should be greater than or equal to 16.

Definition 4.1: For any integer $n \geq 16$, its sextuple (6 – tuple) complete partition can be obtained by taking the parts of n as $\nu = (\nu_1^{m_1} \nu_2^{m_2} ..... \nu_l^{m_l})$ of n such that each integer r, with $6 \leq r \leq n-6$ can be represented at least six different ways as a sum $\sum_{i=1}^{l} \beta_i \nu_i$ with $\beta_i \in \{0, 1, 2, ...m_l\}$.

Theorem 4.2: If a partition $\nu = (\nu_1^{m_1} \nu_2^{m_2} ..... \nu_l^{m_l})$ of a positive integer $n \geq 16$ is a sextuple complete partition then $\nu_{i+1} \leq \sum_{j=1}^{i} m_j \nu_j - 5$ with $i \geq 4$ and ν should have at least two 1's, one 2, one 3, one 4 and one 5 (or) one 1, two 2's,one 3, one 4 and one 5 (or) one 1, one 2, two

3's, one 4 and one 5 (or) one 1, one 2, one 3, two 4's and one 5 (or) one 1, one 2, one 3, one 4 and two 5's as its parts..

Proof : For any integer *n*, its sextuple complete partition can be obtained by taking the value as $n \geq 16$, and the parts of the integer *n* should be equivalent to $(v_1^{m_1} v_2^{m_2} \ldots\ldots v_l^{m_l})$ We can prove this theorem by considering the parts of the integer as $n = v_1^{m_1} v_2^{m_2} v_3^{m_3} v_4^{m_4} v_5^{m_5}$. That is, $n = 1^{m_1} 2^{m_2} 3^{m_3} 4^{m_4} 5^{m_5}$ with $m_1 \geq 2, m_2, m_3, m_4 \ \& \ m_5 \geq 1$ and $v_5 \leq m_1 + m_2 + m_3 + m_4$ is a sextuple complete partition of the integer $n = m_1 v_1 + m_2 v_2 + m_3 v_3 + m_4 v_4 + m_5 v_5$. If it is a sextuple complete partition, then for every integer r, $5 \leq r \leq \sum_{j=1}^{i} m_j v_j - 5$, it can be written as five different ways using the parts 1, 2, 3, 4 and 5. Therefore, $m_5 v_5$, $m_2 v_1 + m_4 v_4$, $m_2 v_2 + m_3 v_3$, $m_2 v_1 + m_2 v_2$ and $m_1 v_1 + m_3 v_3$ are the five representations of *n* with v satisfies the condition $v_{i+1} \leq \sum_{j=1}^{i} m_j v_j - 5.$ Now we check the condition $v_{i+1} \leq \sum_{j=1}^{i} m_j v_j - 5$ for n. Let us assume that

$n = v_1^{m_1} v_2 v_3 v_4 v_5$, $n = v_1 v_2^{m_2} v_3 v_4 v_5$, $n = v_1 v_2 v_3^{m_3} v_4 v_5$, $n = v_1 v_2 v_3 v_4^{m_4} v_5$ and $n = v_1 v_2 v_3 v_4 v_5^{m_5}$ be a sextuple complete partitions of *n* with $m_1 = m_2 = m_3 = m_4 = m_5 = 2$ and $v_1 = 1, v_2 = 2, v_3 = 3, v_4 = 4, v_5 = 5$ ------(1).

Case (i) : If we consider $n = v_1^{m_1} v_2 v_3 v_4 v_5$, to be the sextuple complete partition, it should satisfy the condition

$$v_{i+1} \le \sum_{j=1}^{i} m_j \, v_j - 5 - - - - - - -(2)$$

with $i = 4$. Here $n = 16$ by equation (1) and by equation (2), $5 \le 6$. Therefore, v satisfies the condition.

Case (ii) : If we consider $n = v_1 \, v_2^{m_2} v_3 v_4 v_5$, to be the sextuple complete partition then by equations (1) and (2) $n = 17$ and $5 \le 7$.

Case (iii) : If we consider $n = v_1 v_2 v_3^{m_3} v_4 v_5$, to be the sextuple complete partition then by equations (1) and (2) $n = 18$ and $5 \le 8$.

Case (iv) : If we consider $n = v_1 v_2 v_3 v_4^{m_4} v_5$ to be the sextuple complete partition then by equations (1) and (2) $n = 19$ and $5 \le 9$.

Case (v) : If we consider $n = v_1 v_2 v_3 v_4 v_5^{m_5}$ to be the sextuple complete partition then by equations (1) and (2) $n = 20$ and $5 \le 5$.

If the above cases are true for $n = v_1^{m_1} \, v_2 \, v_3 v_4 v_5$, $n = v_1 \, v_2^{m_2} v_3 v_4 v_5$, $n = v_1 v_2 v_3^{m_3} v_4 v_5$, $n = v_1 v_2 v_3 v_4^{m_4} v_5$ and $n = v_1 v_2 v_3 v_4 v_5^{m_5}$,then the condition $v_{i+1} \le \sum_{j=1}^{i} m_j v_j - 5$ is also true for $v = (v_1^{m_1} v_2^{m_2} \ldots\ldots v_l^{m_l})$. Hence v satisfies the condition $v_{i+1} \le \sum_{j=1}^{i} m_j v_j - 5$.

For a clear understanding numerical illustration is presented below.

Numerical Illustration 4.3: For the integer $n = 22 = 1^2 23456$ its sextuple complete partition is as follows:

Sl. No.	**Integer n**	**Sextuple complete partition of n**
1.	22	$2.1 + 1.2 + 1.3 + 1.4 + 1.5 + 1.6$ $0.1 + 0.2 + 0.3 + 0.4 + 2.5 + 2.6$ $0.1 + 1.2 + 1.3 + 0.4 + 1.5 + 2.6$ $2.1 + 0.2 + 1.3 + 0.4 + 1.5 + 2.6$ $0.1 + 0.2 + 2.3 + 1.4 + 0.5 + 2.6$ $1.1 + 1.2 + 1.3 + 1.4 + 0.5 + 2.6$
2.	23	$1.1 + 0.2 + 0.3 + 0.4 + 2.5 + 2.6$ $1.1 + 1.2 + 1.3 + 0.4 + 1.5 + 2.6$ $1.1 + 0.2 + 2.3 + 1.4 + 0.5 + 2.6$ $2.1 + 1.2 + 1.3 + 1.4 + 0.5 + 2.6$ $0.1 + 1.2 + 1.3 + 2.4 + 2.5 + 0.6$ $0.1 + 2.2 + 2.3 + 2.4 + 1.5 + 0.6$

Corollary 4.4: Let $\nu = (\nu_1^{m_1} \nu_2^{m_2} \ldots\ldots \nu_l^{m_l})$ be a sextuple complete partition of a positive integer n. Then $\nu_{i+1} \leq \sum_{j=1}^{i} 6^{j-1} \nu_j$ where ν_{i+1} is the last part of the sextuple complete partition of an integer.

***Proof*:** In a sextuple complete partition, $n \geq 16$ and $\nu_1, \nu_2, \nu_3, \nu_4$ and ν_5 should be equivalent to 1, 2, 3, 4 and 5 respectively.

$$\nu_{i+1} \leq \nu_1 + \nu_2 + \ldots\ldots + \nu_j \leq 6\nu_1 + 6\nu_2 + \ldots\ldots + 6\nu_j \leq 6^{j-1}\nu_1 + 6^{j-1}\nu_2 + \ldots\ldots + 6^{j-}$$

5. Conclusion

Using the concept of complete partition an attempt has been made at sextuple complete partitions of integers. This work may be extended upto k-tuple complete partitions of integers.

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