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Some Existence Theorems for Periodic Boundary Value Problem of First Order Delay Differential Equation

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Abstract

In this paper we establish existence results for the periodic boundary value problem of first order delay differential equation using Leray - Schauder alternative and Schauder's fixed point theorem. We define lower and upper solutions to establish existence of solution between them. Further we define strict lower and upper solutions for the problem to establish existence of solution strictly between the two.

Keywords: Periodic Boundary Value Problem, Delay Differential Equation, Lower and Upper Solution Mathematics Subject Classification (2010): 35F30, 34K10

1. Introduction

The study of boundary value problems (BVPs) for differential equations with deviating arguments (DEDA) is important because of their applications in various fields such as production problems in Eco- nomics, Biological systems, Physical models etc. Thus, given a DEDA the most natural question that arises is of existence of its solution.

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The existence theory of ordinary differential equations as well as de- lay differential equations (DDE) is well developed in **([1, 4, 12])**. For the fixed-point theory, which is the main source of proving many existence results, one may refer to **[11]**. Existence results on solution between lower and upper solutions for BVP of first order ordinary differential equation are discussed in **[6]**, **[7]**, **[8]** and **[9]**. One may refer **[2]**, **[3]**, **[5]**, **[10]**, **[13]** and **[14]** for BVP of First Order DDEs.

Periodic boundary value problem (PBVP) for first order DDE of the type $x'(t) = f(t, x_t)$, $t \in I = [0, T]$, $x(0) = \phi(0) = x(\phi)(T)$,

$x(\theta) = \phi(\theta)$; $\theta \in [-r, 0]$, is studied in **[1]**. Here $\phi \in C\,[-r, 0]$,

$f : I \times R \to R$ is a continuous function satisfying

$0 \leq f(t, v(t)) + \lambda v(0) - f(t, u(t)) - \lambda u(0) \leq \mu(v(t) - u(t))\ \forall\, u, v \in C[-r, 0]$

such that $u \leq v$ and λ, μ are some constants such that $0 < \mu < \lambda$.

In **[10]**, PBVP for first order DDE $x'(t) = f(t, x, x_t)$, $t \in [0, 2\pi]$,

$x(0) = x(2\pi)$, is investigated using contraction mapping theorem for

operators whose domain and range are different Banach spaces. PBVP for a differential equation with piecewise constant argument (DEPCA) given by $x'(t) = f(t, x, x([t-1]))$, $t \in [0, T]$, $x(0) = x(T)$ is studied in **[14]**. Here $f\colon [0, T] \times R^2 \to R$ is a continuous function satisfying $f(t, u_1, v_1) - f(t, u_2, v_2) \geq -M_1(u_1 - u_2) - M_2(v_1 - v_2)$, for $t \in [0, T]$,

$M_1, M_2 > 0$, $u_1, u_2, v_1, v_2 \in R$ with $\alpha(t) \leq u_2 \leq u_1 \leq \beta(t)$ and

$\alpha([t-1]) \leq v_2 \leq v_1 \leq \beta([t-1])$, where α and β are lower and upper solutions of the given PBVP for DEPCA.

In this paper we consider existence of solution to PBVP for a first Order DDE using Leray - Schauder alternative and Schauder's fixed point theorem.

Let $T > 0$, $\lambda > 0$, $I = [0, T]$, $0 < r < T$ and $f\colon I \times R \to R$ be continuous. We consider the following PBVP for first order DDE:

$$y'(t) + \lambda y(t) = f(t, y(t-r)), \ t \in I, \tag{1}$$

$$y(t) = y(0), \ t \in [-r, 0], \tag{2}$$

$$y(0) = y(T). \tag{3}$$

We use the following notations throughout the paper:

1. $B = C[-r, T] \cap C'[0, T]$, is the set of all functions defined and continuous on $[-r, T]$ and are continuously differentiable on $[0, T]$.
2. $X = \{y \in B: y(t) = y(0), \ \forall \ t \in [-r, 0]\}$.
3. $Y = \{y \in B: y(t) = 0, \forall \ t \in [-r, 0]\}$.
4. $\| x \|_\infty = \sup_{t\in[-r,T]} |x(t)| + \sup_{t\in[0,T]} |\acute{x}(t)|, \forall \ x \in B$
5. $\|(y, \xi\|_* = \|y\|_\infty + |\xi|, \ \forall (y, \xi) \epsilon \ Y \ x \ R$.
6. For a linear operator $L: X \to Y \ x \ R$, $\|L\| = \sup_{\|x\|_\infty=1} \|Lx\|_*$

Note that $(B, \|.\|_\infty), (X, \|.\|_\infty), (Y, \|.\|_\infty)$ and $(Y \times R, \|.\|_*)$ are Banach spaces.

We now state Leray - Schauder alternative [6] and Schauder's fixed point theorem [11].

Theorem 1.1. Leray - Schauder Alternative:

Let C be a complete convex subset of a locally convex Hausdorff linear topological space E and U an open subset of C with $p \in U$. In addition, let $A: U \to C$ be a continuous and compact map. Then either A has a fixed point in U or there is $U \in \partial U$ and $\mu \in (0,1)$ with $u = \mu Au + (1-\mu)p$.

Theorem 1.2. Schauder's Fixed Point Theorem:

Let E be a normed linear space, S: E→E be a continuous and compact map such that S(E) is bounded. Then S has a fixed point.

We state the following Lemma without proof.

Lemma 1.3. Let $L: X \to Y \times \mathbb{R}$ be an operator defined by

$L[y(t)] = (u(t), y(0))$, where:

$$u(t) = \begin{cases} 0, & -r \le t \le 0 \\ y(t) - y(0) + \lambda \int_0^t y(s)ds, & 0 \le t \le T \end{cases}$$

Then

$L: X \to Y \times \mathbb{R}$ is a continuous linear operator.

$L^{-1}: Y \times \mathbb{R} \to X$ exists and it is defined by

$$L^{-1}(u(t), \gamma) = \begin{cases} \gamma, & -r \le t \le 0, \\ \gamma e^{-\lambda t} + \displaystyle\int_0^t e^{-\lambda(t-s)} u'(s)ds, & 0 \le t \le T. \end{cases}$$

$L^{-1}: Y \times \mathbb{R} \to X$ is continuous linear operator.

We also require the following lemma.

Lemma 1.4. Let $f: I \times \mathbb{R} \to \mathbb{R}$ be continuous and $N: X \to Y \times \mathbb{R}$ be defined by $N[y(t)] = (v(t), y(T))$, where

$$v(t) = \begin{cases} 0, & -r \le t \le 0 \\ \int_0^t f(s, y(s-r))ds, & 0 \le t \le T \end{cases}$$

Then, $N: X \to Y \times \mathbb{R}$ is continuous and compact.

Proof. Claim1:-$\Rightarrow$ $y_n \to y$ uniformly on $[-r, T]$ and $y_n' \to y'$ uniformly on $[0, T]$.

Since, $y_n(t) = y_n(0), \forall t \in [-r, 0]$ and $\forall t \in \mathbb{N}$, we have $y(t) = y(0), \forall t \in [-r, 0]$ as $y_n \to y$ pointwise on $[-r, 0]$.

Let $N[y_n(t)] = (v_n(t), y_n(T))$ and $N[y(t)] = (v(t), y(T))$, where

$$v_n(t) = \begin{cases} 0, & -r \le t \le 0, \\ \int_0^t f(s, y_n(s-r))ds, & 0 \le t \le T. \end{cases}$$

and

$$v(t) = \begin{cases} 0, & -r \le t \le 0 \\ \int_0^t f(s, y(s-r))ds, & 0 \le t \le T \end{cases}$$

We show that $Ny_n \to Ny$ in $(Y \times \mathbb{R}, \|.\|\|_*)$.

Since $y_n \to y$ uniformly on $[-r, T], y_n(T) \to y(T)$ in $\mathbb{R}$.

Let $\epsilon > 0$ be arbitrary and $t \in [-r, 0]$. Then $|v_n(t) - v(t)| = 0 < \epsilon, \forall n \in \mathbb{N}$ and $\forall t \in [-r, 0]$.

Let $0 \le t \le T$. Since $y_n \to y$ uniformly on $[-r, T], y_n(s-r) \to y(s-r)$ uniformly on I.

Therefore, $f(s, y_n(s-r)) \to f(s, y(s-r))$ uniformly on I, as $f: I \times \mathbb{R} \to \mathbb{R}$ is continuous.

$\Rightarrow \exists n_0 \in \mathbb{N}$ such that, $|f(s, y_n(s-r)) - f(s, y(s-r))| < \frac{\epsilon}{T}, \forall n \ge n_0$ and $\forall s \in [0, T]$

Therefore, $|v_n(t) - v(t)| \le \int_0^t |f(s, y_n(s-r)) - f(s, y(s-r))| ds$

$$\le \int_0^T |f(s, y_n(s-r)) - f(s, y(s-r))| ds$$

$$< \frac{\epsilon}{T} \int_0^T ds$$

$$= \epsilon.$$

$\Rightarrow |v_n(t) - v(t)| < \epsilon, \forall n \ge n_0$ and $\forall t \in [0, T]$.

$\Rightarrow v_n \to v$ uniformly on $[-r, T]$.

Since $v_n'(t) = f(t, y_n(t-r))$ and $v'(t) = f(t, y(t-r))$, we get

$|v_n'(t) - v'(t)| = |f(t, y_n(t-r)) - f(t, y(t-r))| < \frac{\epsilon}{T}, \forall n \ge n_0$ and $\forall t \in [0, T]$.

$\Rightarrow v_n' \to v'$ uniformly on $[0, T]$.

$\Rightarrow \sup_{t \in [-r,T]} |v_n(t) - v(t)| \to 0$ as $n \to \infty$ and $\sup_{t \in [0,T]} |v_n'(t) - v'(t)| \to 0$ as $n \to \infty$.

$\Rightarrow \|v_n - v\|_\infty \to 0$ as $n \to \infty$.

$\Rightarrow \|Ny_n - Ny\|_* = \|(v_n, y_n(T)) - (v, y(T))\|_*$.

$= \|v_n - v\|_\infty + |y_n(T) - y(T)|$.

$\Rightarrow \|Ny_n - Ny\|_* \to 0$ as $n \to \infty$.

$\Rightarrow N: X \to Y \times \mathbb{R}$ is continuous.

<u>Claim 2</u>:- :- $N: X \to Y \times \mathbb{R}$ is compact.

Let (y_n) be a bounded sequence in $(X, \|\cdot\|_\infty)$.

Therefore, $\exists K > 0$ such that $\|y_n\|_\infty \leq K, \forall n \in \mathbb{N}$.

Let $Ny_n = (v_n, y_n(T)), \forall n \in \mathbb{N}$, where

$$v_n(t) = \begin{cases} 0, & -r \leq t \leq 0, \\ \int_0^t f(s, y_n(s-r))ds, & 0 \leq t \leq T. \end{cases}$$

Since $\|y_n\|_\infty \leq K, \forall n \in \mathbb{N}$, we have

$\sup_{t\in[-r,T]} |y_n(t)| \leq K, \forall n \in \mathbb{N}$ and $\sup_{t\in[0,T]} |y_n'(t)| \leq K, \forall n \in \mathbb{N}$.

Note that $|v_n(t)| = 0; \forall t \in [-r, 0]$.

Also, $\exists \eta > 0$ such that $|f(t, y_n(t-r))| \leq \eta, \forall t \in [0,T]$ and $\forall n \in \mathbb{N}$.

Let $0 \leq t \leq T$. Then,

$|v_n(t)| \leq \int_0^t |f(s, y_n(s-r))| ds \leq \eta \int_0^T ds = \eta T, \forall t \in [0,T]$ and $\forall n \in \mathbb{N}$.

$\Rightarrow \sup_{t\in[-r,T]} |v_n(t)| \leq \eta T, \forall n \in \mathbb{N}$.

Since $v_n'(t) = f(t, y_n(t-r)), \forall n \in \mathbb{N}$, we have $|v_n'(t)| \leq \eta, \forall t \in [0,T]$ and $\forall n \in \mathbb{N}$.

$\Rightarrow \sup_{t\in[0,T]} |v_n'(t)| \leq \eta, \forall n \in \mathbb{N}$.

$\Rightarrow \|v_n\|_\infty \leq \eta T + \eta = (\eta + 1)T, \forall n \in \mathbb{N}$.

$\Rightarrow (v_n)$ is bounded on $(Y, \|\cdot\|_\infty)$.

$\Rightarrow (v_n)$ is uniformly bounded on $[-r, T]$ and (v_n') is uniformly bounded on $[0, T]$.

Let $t_0 \in [-r, T]$ be arbitrary and $\epsilon > 0$ be arbitrary.

CASE-1: Let $t_0 = -r$.

Choose $\delta = \min\left\{\frac{r}{2}, \epsilon\right\}$.Then

$|v_n(t) - v_n(-r)| = 0 < \epsilon, \forall t$ such that $-r < t < -r + \delta$.

$\Rightarrow (v_n)$ is equicontinuous at $-r$.

CASE-2: Let $t_0 = 0$.

Choose $\delta = \min\left\{\frac{r}{2}, \frac{\epsilon}{\eta}\right\}$ and t be such that $|t| < \delta$.

If $t \leq t_0$, then $|v_n(t) - v_n(t_0)| = 0 < \epsilon$.

Let $t \geq t_0$. Then

$$|v_n(t) - v_n(t_0)| = |v_n(t)| \leq \int_0^t f(s, y_n(s-r))ds$$
$$\leq t\eta < \delta\eta \leq \eta \frac{\epsilon}{\eta} = \epsilon$$

$\Rightarrow |v_n(t) - v_n(t_0)| < \epsilon, \forall t$ such that $|t| < \delta$.

$\Rightarrow (v_n)$ is equicontinuous at 0 .

CASE-3: Let $t_0 = T$.

Choose $\delta = \min\left\{\frac{T}{2}, \frac{\epsilon}{\eta}\right\}$ and let $T - \delta < t \leq T$. Then

$$\begin{aligned} |v_n(t) - v_n(t_0)| &= \left|\int_0^t f(s, y_n(s-r))ds - \int_0^T f(s, y_n(s-r))ds\right| \\ &= \left|\int_t^T f(s, y_n(s-r))ds\right| \\ &\leq (T-t)\eta < \delta\eta \leq \eta\frac{\epsilon}{\eta} = \epsilon. \end{aligned}$$

$\Rightarrow |v_n(t) - v_n(t_0)| < \epsilon, \forall t$ such that $T - \delta < t \leq T$.

$\Rightarrow (v_n)$ is equicontinuous at T.

CASE-4: Let $-r < t_0 < 0$.

Choose $\delta = \min\left\{t_0 + \frac{r}{2}, \frac{-t_0}{2}\right\}$. Then

$|v_n(t) - v_n(t_0)| = 0 < \epsilon, \forall t$ such that $|t - t_0| < \delta$.

$\Rightarrow (v_n)$ is equicontinuous at t_0, where $-r < t_0 < 0$.

CASE-5: Let $0 < t_0 < T$.

Choose $\delta = \min\left\{\frac{t_0}{2}, \frac{T-t_0}{2}, \frac{\epsilon}{\eta}\right\}$ and let t be such that $|t - t_0| < \delta$. Then

$$\begin{aligned} |v_n(t) - v_n(t_0)| &= \left| \int_0^t f(s, y_n(s-r))ds - \int_0^{t_0} f(s, y_n(s-r))ds \right| \\ &= \left| \int_{t_0}^t f(s, y_n(s-r))ds \right| \\ &\leq \left| \int_{t_0}^t \left| f(s, y_n(s-r)) \right| ds \right| \\ &\leq \eta |t - t_0| < \delta\eta \leq \eta \frac{\epsilon}{\eta} = \epsilon. \end{aligned}$$

$\Rightarrow |v_n(t) - v_n(t_0)| < \epsilon, \forall t$ such that $|t - t_0| < \delta$.

$\Rightarrow (v_n)$ is equicontinuous at t_0, where $0 < t_0 < T$.

Hence, from CASE-1 to CASE-5, we conclude that (v_n) is equicontinuous on $[-r, T]$.

By Arzela-Ascoli theorem, (v_n) has a subsequence $(v_{(n,m)})$ converging uniformly to v on $[-r, T]$.

Let $(y_{(n,m)})$ be the corresponding subsequence of (y_n).

Since $\|y_n\|_\infty \leq K, \forall n \in \mathbb{N}$, we have $\|y_{(n,m)}\|_\infty \leq K, \forall m \in \mathbb{N}$.

$\Rightarrow (y_{(n,m)})$ is uniformly bounded on $[-r, T]$ and $(y'_{(n,m)})$ is uniformly bounded on $[0, T]$. $\Rightarrow (v_{(n,m)})$ is uniformly bounded on $[-r, T]$ and $(v'_{(n,m)})$ is uniformly bounded on $[0, T]$.

Since $v'_{(n,m)}(t) = f\left(t, y_{(n,m)}(t-r)\right), \forall t \in [0, T]$ and as $f: I \times \mathbb{R} \to \mathbb{R}$ is continuous, $(v'_{(n,m)})$ is equicontinuous on $[0, T]$.

Therefore, by Arzela-Ascoli theorem $(v'_{(n,m)})$ has a subsequence $(v'_{(n,m,q)})$ converging uniformly to u on $[0, T]$.

Since $(v_{(n,m)})$ is a subsequence of (v_n) and $(v_{(n,m,q)})$ is a subsequence of $(v_{(n,m)}), (v_{(n,m,q)})$ is a subsequence of (v_n).

Since $v_{(n,m)} \to v$ uniformly on $[-r, T], v_{(n,m,q)} \to v$ uniformly on $[-r, T]$. $\Rightarrow v_{(n,m,q)} \to v$ uniformly on $[0, T]$.

Since $v'_{(n,m,q)} \to u$ uniformly on $[0, T], u = v'$ on $[0, T]$.

$\Rightarrow v_{(n,m,q)} \to v$ uniformly on $[-r, T]$ as $q \to \infty$ and $v'_{(n,m,q)} \to v'$ uniformly on $[0, T]$ as $q \to \infty$. Hence $\|v_{(n,m,q)} - v\|_{\infty} \to 0$ as $q \to \infty$.

Let $(y_{(n,m,q)})$ be the corresponding subsequence of (y_n).

Now $\|y_n\|_{\infty} \leq K, \forall n \in \mathbb{N}, \Rightarrow \|y_{(n,m,q)}\|_{\infty} \leq K, \forall q \in \mathbb{N}$.

$\Rightarrow |y_{(n,m,q)}(t)| \leq K, \forall q \in \mathbb{N}$ and $\forall t \in [-r, T]$.

$\Rightarrow |y_{(n,m,q)}(T)| \leq K, \forall q \in \mathbb{N}$.

Therefore, by Bolzano-Weierstrass's theorem, $(y_{(n,m,q)}(T))$ has a convergent subsequence $(y_l(T))$ converging to $\gamma \in \mathbb{R}$ as $l \to \infty$.

Let (v_l) be the corresponding subsequence of $(v_{(n,m,q)})$ in $(Y, \|.\|_{\infty})$.

Since, $v_{(n,m,q)} \to v$ in $(Y, \|.\|_{\infty})$ as $q \to \infty$, we have $v_l \to v$ in $(Y, \|.\|_{\infty})$ as $l \to \infty$.

Therefore, $\|Ny_l - (v, \gamma)\|_* = \|(v_l, y_l(T)) - (v, \gamma)\|_*$

$$= \|v_l - v\|_{\infty} + |y_l(T) - \gamma| \to 0 \text{ as } l \to \infty.$$

$\Rightarrow Ny_l \to (v, \gamma)$ in $(Y \times \mathbb{R}, \|.\|_*)$ as $l \to \infty$.

Hence, $N: X \to Y \times \mathbb{R}$ is a compact operator.

2. Existence theorem using Leray-Schauder alternative

In this section, we establish existence of solution for the PBVP (1)-(3) using Leray-Schauder alternative.

Theorem 2.1. Let $f: I \times \mathbb{R} \to \mathbb{R}$ be continuous. If $\exists M > 0$, independent of μ with $\| y \|_{\infty} \neq M$ for any solution y of

$$y'(t) + \lambda y(t) = \mu f(t, y(t-r)), t \in I \tag{4}$$

$$y(t) = y(0), t \in [-r, 0],$$

$$y(0) = y(T)$$

for each $\mu \in (0,1)$, then the PBVP (1) - (3) has at least one solution in X. Proof. Let $L: X \to Y \times \mathbb{R}$ be defined by $L[y(t)] = (u(t), y(0))$, where

$$u(t)=\begin{cases}0, & -r\le t\le 0\\ y(t)-y(0)+\lambda\int_0^t y(s)ds, & 0\le t\le T\end{cases}$$

By Lemma 1.3. $L: X\to Y\times\mathbb{R}$ is linear, bijective and continuous. Also, $L^{-1}: Y\times\mathbb{R}\to X$ exists and is given by

$$L^{-1}(u(t),\gamma)=\begin{cases}\gamma, & -r\le t\le 0,\\ \gamma e^{-\lambda t}+\int_0^t e^{-\lambda(t-s)}u'(s)ds, & 0\le t\le T.\end{cases}$$

By bounded inverse theorem, $L^{-1}: Y\times\mathbb{R}\to X$ is bounded and hence continuous.

Let $N: X\to Y\times\mathbb{R}$ be defined by $N[y(t)]=(v(t),y(T))$, where

$$v(t)=\begin{cases}0, & -r\le t\le 0\\ \int_0^t f(s,y(s-r))ds, & 0\le t\le T\end{cases}$$

Then, by Lemma 1.4, $N: X\to Y\times\mathbb{R}$ is continuous and compact.

Let $S=L^{-1}N: X\to X$. Then

$$S[y(t)]=\begin{cases}y(T), & -r\le t\le 0,\\ y(T)e^{-\lambda t}+\int_0^t e^{-\lambda(t-s)}f(s,y(s-r))ds, & 0\le t\le T.\end{cases}$$

Note that a fixed point of S is a solution of the PBVP (1) - (3).

Let (y_n) be a bounded sequence in $(X,\|\cdot\|_\infty)$. Then since $N: X\to Y\times\mathbb{R}$ is a compact operator, (Ny_n) has a convergent subsequence (Ny_{n_l}) as $l\to\infty$.

Since $L^{-1}: Y\times\mathbb{R}\to X$ is continuous linear operator, $(L^{-1}[Ny_{n_l}])$ is convergent in $(X,\|\cdot\|_\infty)$ as $l\to\infty$.

Therefore (Sy_{n_l}) is convergent in $(X,\|.\|_\infty)$ as $l\to\infty$.

$\Rightarrow S: X\to X$ is a compact operator.

Since composition of continuous operators is continuous, $S: X\to X$ is a continuous operator.

Let $U=\{y\in X: \|y\|_\infty< M\}$. Then U is open in $(X,\|.\|_\infty)$.

By Leray-Schauder alternative either S has a fixed point in $\bar{U}$ or $\exists\mu\in(0,1)$ and $y\in\partial U$ such that $y=\mu Sy$.

Let if possible S have no fixed point in $\bar{U}$. Then $\exists y \in \partial U$ and $\mu \in (0,1)$ such that $y = \mu S y$.

Therefore y is a solution of PBVP (4), (2) and (3).

Since $y \in \partial U$, $\| y \|_\infty = M$.

This is a contradiction to the hypothesis in the statement of the theorem.

Therefore S has fixed point in $\bar{U}$. $\Rightarrow \exists y \in \bar{U}$ such that $y = Sy$.

Hence, y is a solution of PBVP (1) - (3).

3. Existence Theorem using Schauder's fixed point theorem

In this section we define lower and upper solution for PBVP of first order DDE to obtain existence of solution between lower and upper solution. Further we define strict lower and upper solutions for PBVP to obtain existence of solution strictly between strict lower and upper solution. We make use of Schauder's fixed point theorem to obtain our results.

Firstly we have following definitions.

Definition 3.1. A function $\alpha \in B$ is called a lower solution of PBVP (1)(3) if

1. $\alpha'(t) + \lambda\alpha(t) \le f(t, \alpha(t-r)),\ t \in I$.
2. $\alpha(t) \le \alpha(0),\ t \in [-r, 0]$.
3. $\alpha(0) \le \alpha(T)$.

A function $\beta \in B$ is called an upper solution of PBVP (1)- (3) if the above inequalities are reversed.

Definition 3.2. A lower solution $\alpha \in B$ of PBVP (1)- (3), is called strict lower solution of the PBVP (1)- (3) if $\alpha'(t) + \lambda\alpha(t) < f(t, \alpha(t-r)),\ t \in I$.

An upper solution $\beta \in B$ of PBVP (1)- (3), is called strict upper solution of PBVP (1)- (3) if the above inequality is reversed.

We now prove the existence of atleast one solution to the PBVP between the lower and upper solutions.

Theorem 3.3. Let $f: I \times \mathbb{R} \to \mathbb{R}$ be continuous and $f(t,x) \leq f(t,y)$, $\forall (t,x),(t,y) \in I \times \mathbb{R}$ such that $x \leq y$. Let $\alpha, \beta \in X$ be lower and upper solutions of PBVP (1)- (3) such that $\alpha(t) \leq \beta(t) \forall t \in [-r,T]$. Then, the $PBVP$ (1)- (3) has atleast one solution $y \in X$ such that

$\alpha(t) \leq y(t) \leq \beta(t), \forall t \in [-r,T]$.

Proof.

Since $\alpha, \beta \in X$ are lower and upper solutions of PBVP (1)-(3), we have

1. $\alpha'(t) \leq f(t, \alpha(t-r)),\ t \in I$.
2. $\beta'(t) \geq f(t, \beta(t-r)),\ t \in I$.
3. $\alpha(t) \leq \alpha(0),\ t \in [-r,0]$.
4. $\beta(t) \geq \beta(0),\ t \in [-r,0]$.
5. $\alpha(0) \leq \alpha(T)$.
6. $\beta(0) \geq \beta(T)$.

Further, since $\alpha, \beta \in X, \alpha(t) = \alpha(0), \forall t \in [-r,0]$ and

$\beta(t) = \beta(0), \forall t \in [-r,0]$.

Let $p: I \times \mathbb{R} \to \mathbb{R}$ be defined by,

$p(t,x) = \max\{\alpha(t), \min\{x, \beta(t)\}\}, \forall (t,x) \in I \times \mathbb{R}$.

Since $\alpha(t) \leq \max\{\alpha(t), \{x, \beta(t)\}\}, \alpha(t) \leq p(t,x), \forall (t,x) \in I \times \mathbb{R}$.

Since $\alpha(t) \leq \beta(t), t \in [-r,T], \alpha(t) \leq \beta(t), t \in I$.

Also $\min\{x, \beta(t)\} \leq \beta(t), \forall (t,x) \in I \times \mathbb{R}$.

$\Rightarrow \max\{\alpha(t), \min\{x, \beta(t)\}\} \leq \beta(t), \forall (t,x) \in I \times \mathbb{R}$.

$\Rightarrow p(t,x) \leq \beta(t), \forall (t,x) \in I \times \mathbb{R}$.

$\Rightarrow \alpha(t) \leq p(t,x) \leq \beta(t), \forall (t,x) \in I \times \mathbb{R}$.

Hence $p: I \times \mathbb{R} \to \mathbb{R}$ is continuous and bounded.

Let $F^*: I \times \mathbb{R} \to \mathbb{R}$ be defined by

$$F^*(t,y)=\begin{cases} f(t,\alpha(t-r)), & y\le \alpha(t-r),\\ f(t,y), & \alpha(t-r)\le y\le \beta(t-r),\\ f(t,\beta(t-r)), & y\ge \beta(t-r).\end{cases}$$

Then $F^*: I\times\mathbb{R}\to\mathbb{R}$ is continuous and bounded.

Let $L: X\to Y\times\mathbb{R}$ be defined by $L[y(t)]=(u(t),y(0))$, where

$$u(t)=\begin{cases} 0, & -r\le t\le 0\\ y(t)-y(0)+\lambda\int_0^t y(s)ds, & 0\le t\le T\end{cases}$$

Therefore, by Lemma 1.3. $L: X\to Y\times\mathbb{R}$ is linear, bijective and continuous.

Also, $L^{-1}: Y\times\mathbb{R}\to X$ exists and is given by

$$L^{-1}(u(t),\gamma)=\begin{cases} \gamma, & -r\le t\le 0\\ \gamma e^{-\lambda t}+\int_0^t e^{-\lambda(t-s)}u'(s)ds, & 0\le t\le T\end{cases}$$

By bounded inverse theorem, $L^{-1}: Y\times\mathbb{R}\to X$ is bounded and hence continuous.

Let $N: X\to Y\times\mathbb{R}$ be defined as, $N[y(t)]=(v(t),p(T,y(T)))$, where

$$v(t)=\begin{cases} 0, & -r\le t\le 0,\\ \int_0^t F^*(s,y(s-r))ds, & 0\le t\le T.\end{cases}$$

Therefore, $N: X\to Y\times\mathbb{R}$ is compact and continuous.

Let, $S=L^{-1}N: X\to X$. Then,

$$S[y(t)]=\begin{cases} p(T,y(T)), & -r\le t\le 0,\\ p(T,y(T))e^{-\lambda t}+\int_0^t e^{-\lambda(t-s)}F^*(s,y(s-r))ds, & 0\le t\le T.\end{cases}$$

Then, a fixed point of $S: X\to X$ is a solution of

$$y'(t)+\lambda y(t)=F^*(t,y(t-r)), t\in I,$$
$$y(t)=y(0), t\in[-r,0], \quad (5)$$
$$y(0)=p(T,y(T)). \quad (6)$$

Since, $L^{-1}: Y\times\mathbb{R}\to X$ is continuous linear operator and $N: X\to Y\times\mathbb{R}$ is compact, therefore $S: X\to X$ is a compact operator.

Also, $S: X \to X$ to be continuous, as composition of continuous operators is continuous.

Further, p and F^* are bounded $\Rightarrow S(X)$ is a bounded subset of $(X, \|.\|_\infty)$. Therefore, by Schauder's fixed point theorem, $S: X \to X$ has a fixed point $y \in X$.

Therefore, y is a solution of BVP (5), (2) and (6).

We show that $\alpha(t) \leq y(t) \leq \beta(t), \forall t \in [-r, T]$.

Since $\alpha(0) \leq \alpha(T) \leq p(T, y(T)) \leq \beta(T) \leq \beta(0)$,

we have $\alpha(0) \leq p(T, y(T)) \leq \beta(0)$.

$\Rightarrow \alpha(0) \leq y(0) \leq \beta(0)$.

$\Rightarrow \alpha(t) \leq y(t) \leq \beta(t), \forall t \in [-r, 0]$.

Therefore, it is enough to show that $\alpha(t) \leq y(t) \leq \beta(t), \forall t \in (0, T]$.

Let $\epsilon > 0$ be arbitrary.

Define, $\alpha_\epsilon(t) = \alpha(t) - \epsilon(t + 2r), \forall t \in [-r, T]$ and

$\beta_\epsilon(t) = \beta(t) + \epsilon(t + 2r), \forall t \in [-r, T]$.

$\Rightarrow \alpha_\epsilon(t) < \alpha(t), \forall t \in [-r, T]$ and $\beta_\epsilon(t) > \beta(t), \forall t \in [-r, T]$.

Also, $\alpha'_\epsilon(t) = \alpha'(t) - \epsilon, \forall t \in [0, T]$ and $\beta'_\epsilon(t) = \beta'(t) + \epsilon, \forall t \in [0, T]$.

$\Rightarrow \alpha'_\epsilon(t) < \alpha'(t), \forall t \in [0, T]$ and $\beta'_\epsilon(t) > \beta'(t), \forall t \in [0, T]$.

Since $\alpha(0) \leq y(0) \leq \beta(0)$, we have $\alpha_\epsilon(0) < \alpha(0) \leq y(0) \leq \beta(0) < \beta_\epsilon(0)$. $\Rightarrow \alpha_\epsilon(0) < y(0) < \beta_\epsilon(0)$.

Therefore, we prove that $\alpha_\epsilon(t) < y(t) < \beta_\epsilon(t), \forall t \in (0, T]$.

Let, if possible $\exists t_1 \in (0, T]$ such that

$\alpha_\epsilon(t) < y(t) < \beta_\epsilon(t), \forall t \in (0, t_1)$ and $y(t_1) = \beta_\epsilon(t_1)$.

Therefore, $y(t_1) > \beta(t_1)$.

Also, $y(t_1 - h) < \beta_\epsilon(t_1 - h)$ for sufficiently small $h > 0$.

$\Rightarrow \frac{\beta_\epsilon(t_1) - \beta_\epsilon(t_1 - h)}{h} < \frac{y(t_1) - y(t_1 - h)}{h}. \Rightarrow \beta'_\epsilon(t_1) \leq y'(t_1)$

CASE-1: Let $0 < t_1 \leq r$.

$\Rightarrow -r < t_1 - r \leq 0$.

$\Rightarrow \alpha(t_1 - r) \leq y(t_1 - r) \leq \beta(t_1 - r)$.

Therefore, $y'(t_1) = F^*\big(t_1, y(t_1 - r)\big) - \lambda y(t_1)$

$$= f\big(t_1, y(t_1 - r)\big) - \lambda y(t_1)$$
$$\leq f\big(t_1, \beta(t_1 - r)\big) - \lambda \beta(t_1)$$
$$\leq \beta'(t_1)$$
$$< \beta_\varepsilon'(t_1), \text{ which is a contradiction.}$$

CASE-2: Let $r < t_1 \leq T$.

$\Rightarrow 0 < t_1 - r \leq t_1$.

$\Rightarrow \alpha_\varepsilon(t_1 - r) < y(t_1 - r) < \beta_\varepsilon(t_1 - r)$.

CASE-2.1: Let $y(t_1 - r) < \alpha(t_1 - r)$.
Therefore, $y'(t_1) = F^*\big(t_1, y(t_1 - r)\big) - \lambda y(t_1)$

$$= f\big(t_1, \alpha(t_1 - r)\big) - \lambda y(t_1)$$
$$\leq f\big(t_1, \beta(t_1 - r)\big) - \lambda y(t_1)$$
$$< f\big(t_1, \beta(t_1 - r)\big) - \lambda \beta(t_1)$$
$$\leq \beta'(t_1)$$
$$< \beta_\varepsilon'(t_1), \text{ which is a contradiction.}$$

CASE-2.2: Let $\alpha(t_1 - r) \leq y(t_1 - r) \leq \beta(t_1 - r)$.
Therefore, $y'(t_1) = F^*\big(t_1, y(t_1 - r)\big) - \lambda y(t_1)$

$$= f\big(t_1, y(t_1 - r)\big) - \lambda y(t_1)$$
$$\leq f\big(t_1, \beta(t_1 - r)\big) - \lambda \beta(t_1)$$
$$\leq \beta'(t_1)$$
$$< \beta_\varepsilon'(t_1), \text{ which is a contradiction.}$$

CASE-2.3: Let $y(t_1 - r) > \beta(t_1 - r)$.
Therefore, $y'(t_1) = F^*\big(t_1, y(t_1 - r)\big) - \lambda y(t_1)$

$$= f\big(t_1, \beta(t_1 - r)\big) - \lambda y(t_1)$$
$$< f\big(t_1, \beta(t_1 - r)\big) - \lambda \beta(t_1)$$
$$\leq \beta'(t_1)$$
$$< \beta_\varepsilon'(t_1), \text{ which is a contradiction.}$$

Hence, from the above discussed cases we conclude that

$\alpha_\epsilon(t) < y(t) < \beta_\epsilon(t), \forall\, t \in [0, T]$ and $\forall \epsilon > 0$.

Therefore by taking $\epsilon \to 0$ we get, $\alpha(t) \le y(t) \le \beta(t), \forall\, t \in [0, T]$.

$\Rightarrow \alpha(t) \le y(t) \le \beta(t), \forall\, t \in [-r, T]$.

Therefore, $F^\star(t, y(t-r)) = f(t, y(t-r)), \forall\, t \in [0, T]$ and $p(T, y(T)) = y(T)$. Hence, $y \in X$ is a solution of **(1) - (3)** such that

$\alpha(t) \le y(t) \le \beta(t), \forall\, t \in [-r, T]$. □

Theorem 3.4. *Let* $f: I \times R \to R$ *be continuous and* $f(t, x) \le f(t, y)$, $\forall\, (t, x), (t, y) \in I \times R$ *such that* $x \le y$. *Let* $\alpha, \beta \in X$ *be strict lower and upper solutions of* ***(1) - (3)*** *such that* $\alpha(t) \le \beta(t), \forall\, t \in [-r, T]$. *Then, PBVP* ***(1) - (3)*** *has atleast one solution* $y \in X$ *such that* $\alpha(t) \le y(t) \le \beta(t)$, $\forall\, t \in [-r, 0]$ *and* $\alpha(t) < y(t) < \beta(t), \forall\, t \in (0, T]$.

Proof.

Since $\alpha, \beta \in X$ are strict lower and upper solutions of **(1) - (3)**, we have

1. $\alpha'(t) < f(t, \alpha(t - r)),\ t \in I$.
2. $\beta'(t) > f(t, \beta(t - r)),\ t \in I$.
3. $\alpha(t) \le \alpha(0),\ t \in [-r, 0]$.
4. $\beta(t) \ge \beta(0),\ t \in [-r, 0]$.
5. $\alpha(t) \le \beta(t), t \in [-r, T]$.
6. $\alpha(0) \le \alpha(T)$.
7. $\beta(0) \ge \beta(T)$.

As $\alpha, \beta \in X, \alpha(t) = \alpha(0), \forall\, t \in [-r, 0]$ and $\beta(t) = \beta(0), \forall\, t \in [-r, 0]$. Let $p, F^\star, L, N$ and S be as defined in Theorem 3.3.

Therefore, by Schauder's fixed point theorem, $S: X \to X$ has a fixed point $y \in X$.

This will imply that y is a solution of **(5), (2)** and **(6)**.

Since $\alpha(0) \le \alpha(T) \le p(T, y(T)) \le \beta(T) \le \beta(0)$, therefore $\alpha(0) \le p(T, y(T)) \le \beta(0)$.

$\Rightarrow \alpha(0) \le y(0) \le \beta(0)$.

$\Rightarrow \alpha(t) \le y(t) \le \beta(t), \forall\, t \in [-r, 0]$.

Next, we show that $\alpha(t) < y(t) < \beta(t), \forall\, t \in (0, T]$. Let, if possible $\exists\, t_1 \in (0, T]$ such that

$\alpha(t) < y(t) < \beta(t), \forall\, t \in (0, t_1)$ and $y(t_1) = \beta(t_1)$.

$\Rightarrow y(t_1 - h) < \beta(t_1 - h)$ for sufficiently small $h > 0$.

$\frac{\beta(t_1)-\beta(t_1-h)}{h} < \frac{y(t_1) - y(t_1-h)}{h}$

$\Rightarrow \beta'(t_1) \le y'(t_1)$.

<u>CASE-1</u>: Let $0 < t_1 \le r$.

$\Rightarrow -r < t_1 - r \le 0$.
$\Rightarrow \alpha(t_1 - r) \le y(t_1 - r) \le \beta(t_1 - r)$.

Therefore, $y'(t_1) = F^{\star}(t_1, y(t_1 - r)) - \lambda y(t_1)$
$= f(t_1, y(t_1 - r)) - \lambda y(t_1)$
$\le f(t_1, \beta(t_1 - r)) - \lambda \beta(t_1)$
$< \beta'(t_1)$, which is a contradiction. <u>CASE-2</u>: Let $r < t_1 \le T$.
$\Rightarrow 0 < t_1 - r \le t_1$.
$\Rightarrow \alpha(t_1 - r) < y(t_1 - r) < \beta(t_1 - r)$.

Therefore, $y'(t_1) = F^{\star}(t_1, y(t_1 - r)) - \lambda y(t_1)$
$= f(t_1, y(t_1 - r)) - \lambda y(t_1)$
$\le f(t_1, \beta(t_1 - r)) - \lambda \beta(t_1)$
$< \beta'(t_1)$, which is a contradiction.

Hence, from CASE-1 & CASE-2 we have, $\alpha(t) < y(t) < \beta(t), \forall\, t \in (0, T]$. Therefore, $F^{\star}(t, y(t - r)) = f(t, y(t - r)), \forall\, t \in [0, T]$ and $p(T, y(T)) = y(T)$. Hence, $y \in X$ is a solution of PBVP for FODDE (1) - (3) such that

$\alpha(t) \le y(t) \le \beta(t), \forall\, t \in [-r, 0]$ and $\alpha(t) < y(t) < \beta(t), \forall\, t \in (0, T]$. □

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