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Eulerian and Unicursal Soft Graphs

Jinta Jose* , Bobin George† and Rajesh K. Thumbakara‡

Abstract

In 1999, Molodtsov proposed the use of soft sets as a means of managing imprecision and uncertainty in decision making. This methodology provides a means of modelling vague and uncertain information. Thumbakara and George introduced the notion of soft graphs, which offer a parameterized perspective on graphs. This study presents Eulerian and unicursal soft graphs and explores some of their characteristics.

Keywords: Soft Graph, Eulerian Soft Graph, Unicursal Soft Graph

Mathematics Subject Classification 2020*:* 05C99

1. Introduction

Graph theory is an intriguing and valuable area of discrete mathematics that was first introduced by Leonhard Euler in the 18th century. Soft set theory, an innovative mathematical technique for managing uncertainties, was presented by Molodtsov [8] in 1999 and has proven useful in solving practical problems. The idea of soft sets has been studied extensively by Maji *et. al*. in their works [6,7]. They have utilized this concept in a variety of decision-making contexts. In 2014, Thumbakara and George introduced the concept of soft graphs and explored it further [12,13]. They also introduced related operations and concepts such as soft semigraphs [4] and soft hypergraphs [3]. Other researchers have expanded upon this

* Science and Humanities Department, VJCET, Vazhakulam, India; jinta@vjcet.org

† Mathematics Department, Pavanatma College, Murickassery, India; bobingeorge@pavanatmacollege.org

‡ Mathematics Department, M.A. College (Autonomous), Kothamangalam, India; rthumbakara@macollege.in

work, including Akram and Nawas [1, 2], who defined soft trees, soft bridges, soft cut vertices, and soft cycles in 2015. Thenge *et. al*. [10, 11] made contributions to connected soft graphs, examining concepts such as radius, diameter, centre, and degree. In 2019, Manju and Sarala studied the concept of soft bipartite graphs [9]. Baghernejad and Borzooei [14] explained more concepts in soft graphs. Currently, Euler graphs have achieved significant heights in various fields like computer science, physical science, communication science, economics, and more. This paper introduces the concept of Eulerian and unicursal soft graphs and investigates some of their properties.

2. Preliminaries

2.1. Soft Graph

[1] Akram and Nawas redefined a soft graph as follows: "Let $F^*= (X, D)$ be a simple graph and T be any non-empty set. Let R' be an arbitrary relation between elements of T and elements of X. That is $R' \subseteq T \times X$. A mapping $R: T \to P(X)$ can be defined as $R(t)=\{y \in X | tR'y\}$. Also, define $S: T \to P(D)$ by $S(t)=\{uv \in D | \{u,v\} \subseteq R(t)\}$. The pair (R, T) is a soft set over X and the pair (S, T) is a soft set over D. Then, the 4-tuple $F = (F^*, R, S, T)$ is called a soft graph if it satisfies the following conditions:

1) $F^*= (X, D)$ is a simple graph,

2) T is a nonempty set of parameters,

3) (R, T) is a soft set over X,

4) (S, T) is a soft set over D,

5) $(R(t), S(t))$ is a subgraph of F^* for all $t \in T$

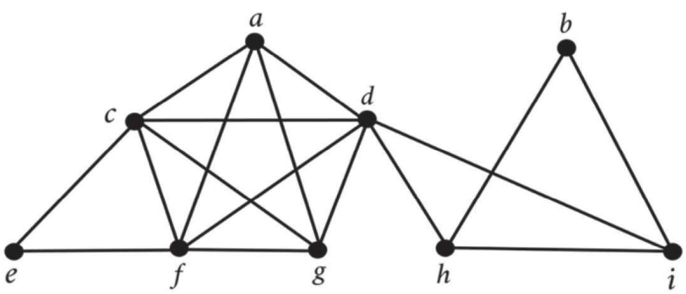
If we represent $(R(t), S(t))$ by $H(t)$ then, the soft graph F is also given by $\{H(t): t \in T\}$. $H(t)$ corresponding to a parameter t in T is called a part of the soft graph F. The part degree of the vertex v in $H(t)$ denoted by $d(v)[H(t)]$ is the degree of the vertex v in that part $H(t)$." Baghernejad and Borzooei [14] defined the concept of degrees related to soft graphs as follows: "Let $W = U_{t \in T} R(t)$ be the vertex set of the soft graph F. Then the degree of $v \in W$ is denoted by $d_F(v)$ and is defined by $d_F(v)=\Sigma_{t\in T}\, d(v)[H(t)]$. If $v\in X$ and v is not in W then $d_F(v)=0$. The vertex degree of F is denoted by $D(F)$ and is defined by $D(F)= \Sigma_{v\in W}\, d_F(v)$."

3. Eulerian Soft Graphs

Definition 3.1. Consider a soft graph $F = (F^*, R, S, T)$, where F^* is given by $\{H(t): t\in T\}$. In this context, any part $H(t)$ of F is termed Eulerian if it contains an Euler tour. In other words, $H(t)$ encompasses a closed trail that traverses every edge within $H(t)$.

Definition 3.2. A soft graph $F = (F^*, R, S, T)$ derived from a graph F^* is deemed Eulerian if each part $H(t)$ of F is Eulerian.

Example 3.1. Consider a graph $F^*= (X, D)$ given in *Fig. 1*.



$T = \{g, b\}$ be a parameter set. Define a mapping $R: T\rightarrow P(X)$ by $R(t) = \{y\in X|t\, Ry\Leftrightarrow d(t,y) \leq 1\}$ for all $t \in T$. That is, $R(g) = \{a, c, d, f, g\}$ and $R(b) = \{b, h, i\}$. Let $S: T\rightarrow P(D)$ be another function defined by $S(t) = \{uv\in D|\ \{u, v\} \subseteq R(t)\}$ for all $t \in T$. That is, $S(g) = \{ac, ad, af, ag, cd, cf, cg, df, dg, fg\}$ and $S(b) = \{bh, bi, hi\}$. Here (R, T) and (S, T) are soft sets over X and D respectively. Thus $H(g) = (R(g), S(g))$ and $H(b) = (R(b), S(b))$ are subgraphs of F^* and is given in *Fig. 2*. Hence, $F= \{H(g), H(b)\}$ is a soft graph of F^*.

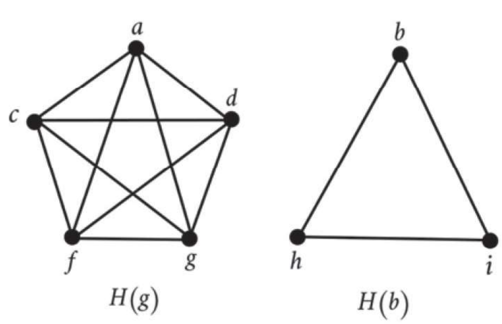


Fig. 2. F = {H(g), H(b)}

Table 1 provides the tabulated depiction of this soft graph.

T/X	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*
g	1	0	1	1	0	1	1	0	0
b	0	1	0	0	0	0	0	1	1

T/D	ac	ad	af	ag	bh	bi	ce	cd	cf	cg	df	dg	dh	di	ef	fg	hi
g	1	1	1	1	0	0	0	1	1	1	1	1	0	0	0	1	0
b	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	1

Table 1

Here the soft graph *F* has 2 parts *H(g)* and *H(b)*. In *H(g)*, T_1 = *agcdfadgfca* is an Euler tour and so *H(g)* is an Eulerian part. Also, in *H(b)*, T_2 = *bhib* is an Euler tour and so *H(b)* is an Eulerian part. Hence, *F* is an Eulerian soft graph.

Example 3.2. Consider a graph *F*= (X, D)* given in *Fig. 3*.

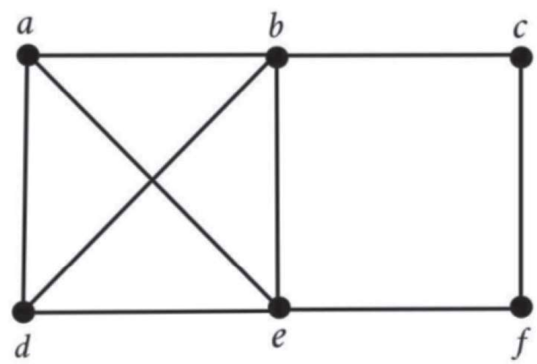


Fig. 3. F*= (X, D)

Let *T*= {*a, c*} be a parameter set. Define a mapping *R: T→ P(X)* by $R(t)=\{y\in X|tRy\Leftrightarrow d(t,y)\leq 1\}$ for all $t\in T$. That is, *R(a) = {a, b, d, e}* and *R(c) = {b, c, f}*.

Define another mapping *S: T→P(D)* by *S(t)* = $\{uv\in D|\{u,v\}\subseteq R(t)\}$ for all $t\in T$. That is, *S(a) = {ab, ad, ae, bd, be, de*} and *S(c) = {bc, cf*}. Here *(R, T)* and *(S, T)* are soft sets over *X* and *D* respectively. Thus *H(a) = (R(a), S(a))* and *H(c) = (R(c), S(c))* are subgraphs of *F** and is shown in *Fig. 4.* So, *F= {H(a), H(c)}* is a soft graph of *F**.

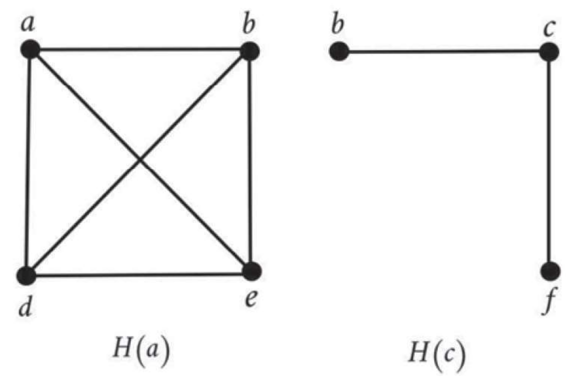


Fig. 4. F = {H(a), H(c)}

Table 2 provides the tabulated depiction of this soft graph.

T/X	*a*	*b*	*c*	*d*	*e*	*f*
a	1	1	0	1	1	0
c	0	1	1	0	0	1

T/D	*ab*	*ad*	*ae*	*bc*	*bd*	*be*	*cf*	*de*	*ef*
a	1	1	1	0	1	1	0	1	0
c	0	0	0	1	0	0	1	0	0

Table 2

Here *F* has *H(a)* and *H(c)* as parts. In *H(a)* and *H(c)*, there exists no Euler tour and so *H(a)* and *H(c)* are not Eulerian parts. That is, all parts of *F* are not Eulerian and so *F is* not an Eulerian soft graph.

Remark 3.1. A soft graph *F = (F*, R, S, T)* of an Eulerian graph *F*= (X, D)* need not be an Eulerian soft graph.

Example 3.3. Consider an Eulerian graph *F*= (X, D)* given in *fig. 5*.

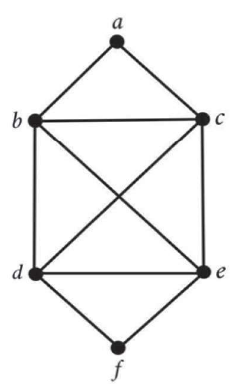


Fig. 5. F*= (X, D)

Let *T*= {*d*, *f*} be a parameter set. Define a mapping *R: T→ P(X)* by $R(t)=\{y\in X|tRy\Leftrightarrow d(t,y)\leq 1\}$ for all $t\in T$. That is, *R(d) = {b, c, d, e, f}* and *R(f) = {d, e, f}*

Also define another mapping *S: T→P(D)* by $S(t)=\{uv\in D\,|\,\{u,v\}\subseteq R(t)\}$ for all $t\in T$. That is, *S(d) = {bc, bd, be, cd, ce, de, df, ef}* and *S(f) = {de, df, ef}*. Here *(R, T)* and *(S, T)* are soft sets over *X* and *D* respectively. Thus *H(d) = (R(d), S(d))* and *H(f) = (R(f), S(f))* are subgraphs of *F** and is shown in *Fig. 6*. So, *F= {H(d), H(f)}* is a soft graph of *F**.

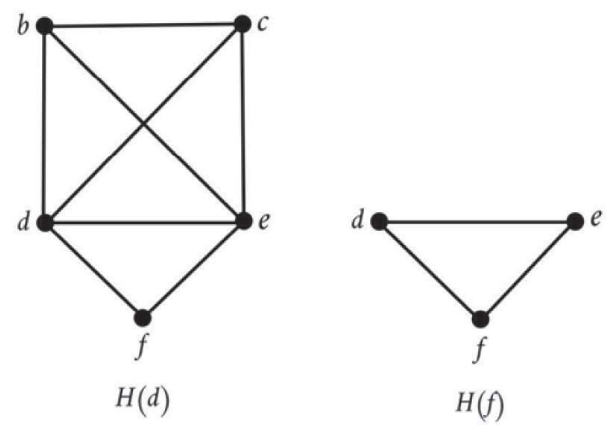


Fig. 6. F = {H(d), H(f)}

Table 3 provides the tabulated depiction of this soft graph.

T/X	*a*	*b*	*c*	*d*	*e*	*f*
d	0	1	1	1	1	1
f	0	0	0	1	1	1

T/D	*ab*	*ac*	*bc*	*bd*	*be*	*cd*	*ce*	*de*	*df*	*ef*
d	0	0	1	1	1	1	1	1	1	1
f	0	0	0	0	0	0	0	1	1	1

Table 3

Here *F has H(d)* and *H(f)* as parts. In *H(d)*, there is no Euler tour and so *H(d)* is not an Eulerian part. In *H(f)*, T_1 = *defd* is an Euler tour and so *H(f)* is an Eulerian part. That is, all parts of *F* are not Eulerian and so *F is* not an Eulerian soft graph.

Remark 3.2. A soft graph formed from an Eulerian graph may be an Eulerian soft graphs in some cases. The following example is such a case.

Example 3.4. Consider an Eulerian graph *F*= (X, D)* given in *Fig. 7*.

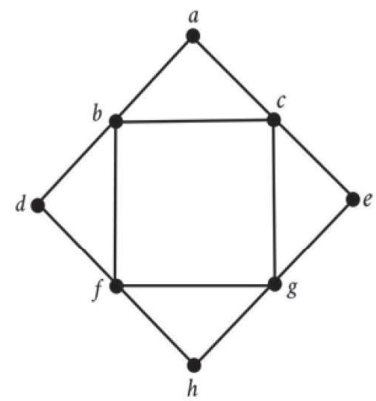


Fig. 7. F*= (X, D)

Let *T*= {*d, c*} be a parameter set. Define a mapping *R: T→ P (X)* by $R(t)=\{y\in X|tRy\Leftrightarrow d(t,y)\leq 1\}$ for all $t\in T$. That is, *R(d) = {b, d, f}* and *R(c) = {a, b, c, e, g}*. Define another mapping *S: T→P (D)* by *S(t)* = $\{uv\in D|\{u,v\}\subseteq R(t)\}$ for all $t\in T$. That is, *S(d) = {bd, bf, df}* and *S(c) = {ab, ac, bc, ce, cg, eg}*. Here *(R, T)* and *(S, T)* are soft sets over *X* and *D* respectively. Thus *H(d) = (R(d), S(d))* and *H(c) = (R(c), S(c))* are subgraphs of *F** and is shown in *Fig. 8*. So, *F= {H(d), H(c)}* is a soft graph of *F**.

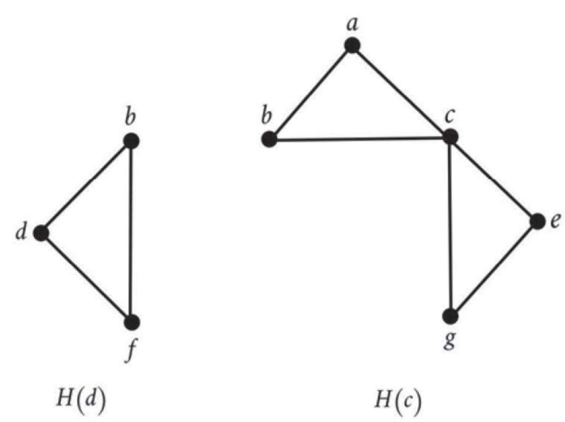


Fig. 8. F = {H(d), H(c)}

Table 4 provides the tabulated depiction of this soft graph.

T/X	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*
d	0	1	0	1	0	1	0	0
c	1	1	1	0	1	0	1	0

T/D	*ab*	*ac*	*bc*	*bd*	*bf*	*ce*	*cg*	*df*	*eg*	*fg*	*fh*	*gh*
d	0	0	0	1	1	0	0	1	0	0	0	0
c	1	1	1	0	0	1	1	0	1	0	0	0

Table 4

Here F has $H(d)$ and $H(c)$ as parts. In $H(d)$, $T_1 = bdfb$ is an Euler tour and so $H(d)$ is an Eulerian part. In $H(c)$, $T_2 = abcgeca$ is an Euler tour and so $H(c)$ is an also Eulerian part. That is, all parts of F are Eulerian and so F is an Eulerian soft graph.

Theorem 3.1. Let $F^*= (X, D)$ be an Eulerian graph. Then F^* is an Eulerian soft graph of itself.

Proof: Let $F^*= (X, D)$ be the given Eulerian graph having vertex set $X=\{v_1, v_2, \ldots, v_n\}$. Then F^* contains an Euler tour and F^* is a connected graph. Let the parameter set be T and suppose T contains any one of the vertices $v_1, v_2, \ldots, v_n$, say v_i. That is, $T=\{v_i\}$. Define a mapping $R: T\to P(X)$ by $R(t)=\{y\in X \mid tRy \Leftrightarrow d(t,y)\leq n\}$ for all $t \in T$. Since F^* is connected and contains only n vertices $R(v_i)$ will contain all vertices of F^*. That is, $R(v_i) = \{v_1, v_2, \ldots, v_n\}$. Also, define another mapping $S: T\to P(D)$ by $S(t)= \{uv\in D \mid \{u,v\}\subseteq R(t)\}$, for all $t \in T$. Then $S(v_i)$ will contain all edges of F^*. That is, the soft graph $F = (F^*, R, S, T)$ which is also represented by $\{H(v_i)\}$ contains only one part $H(v_i)= (R(v_i), S(v_i))$, where $R(v_i)=X$ and $S(v_i)= D$. That is, the soft graph $F= \{H(v_i)\}$ is F^* itself where F^* is the given Eulerian graph. Hence the Eulerian graph F^* is the Eulerian soft graph of itself.

Example 3.5. Consider an Eulerian graph $F^*= (X, D)$ given in *Fig. 5* which contains 6 vertices a, b, c, d, e, and f.

Let $T= \{c\}$ be a parameter set. Define a mapping $R: T\to P(X)$ by $R(t)=\{y\in X \mid tRy \Leftrightarrow d(t,y)\leq 6\}$ for all $t \in T$. That is, $R(c) = \{a, b, c, d, e, f\}$.

Also define another mapping $S: T\to P(D)$ by $S(t) = \{uv\in D \mid \{u,v\}\subseteq R(t)\}$ for all $t \in T$. That is, $S(c) = \{ab, ac, bc, bd, be, cd, ce, de, df, ef\}$. Here (R, T) and (S, T) are soft sets over X and D respectively. Thus $H(c) = (R(c), S(c))$ is a subgraph of F^* (Clearly $H(c)$ is F^* itself). Hence, $F= (F^*, R, S, T)$ is a soft graph of F^* and is the same as F^*. Hence $F= F^*$ is an Eulerian soft graph of F^*.

Remark 3.3. Let $F^*= (X, D)$ be a graph and $F = (F^*, R, S, T)$ be an Eulerian soft graph of F^*. Then F^* may not be an Eulerian graph. This is evident from Example 3.1.

Theorem 3.2. Let $F = (F^*, R, S, T)$ be a soft graph of $F^*= (X, D)$ which is represented by $\{H(t): t \in T\}$. Assume that in a part $H(t)$ of F, the part degree of every vertex is at least two. Then, the part $H(t)$ contains a cycle.

Proof. Let $H(t) = (R(t), S(t))$ be a part of F for some $t \in T$. Assume that $d(v)[H(t)] \geq 2, \forall v \in R(t)$. Let v_1 be any vertex in $H(t)$. As the degree of vertex v_1 in the part $H(t)$ is greater than or equal to 2, it implies the existence of an edge, denoted as e_1, where one end connects to v_1 and the other end connects to v_2 (for instance), with v_2 being distinct from v_1. We go through the same procedure within the part $H(t)$ of F. As $H(t)$ consists of a limited number of vertices, there will come a point where we select a vertex v_m that has already been selected earlier. When v_m is the first repeated vertex, the path between its first and second occurrences in $H(t)$ will form a cycle in $H(t)$.

Theorem 3.3. Let $F = (F^*, R, S, T)$ be a soft graph of $F^*= (X, D)$ in which all parts are connected components. Then F *is* an Eulerian soft graph if and only if part degree $d(v)[H(t)]$ of a vertex v is even for any vertex v in F and in every part $H(t)$ of F containing v.

Proof. Suppose F *is* a soft graph that satisfies the conditions for being Eulerian. In such a graph, all parts $H(t)$ will also be Eulerian. For any such Eulerian part $H(t)$, there exists an Euler tour T in $H(t)$ that starts and ends at a vertex u. If there is any other vertex v in $H(t)$ apart from u, then it must be included in the tour T as $H(t)$ is a connected graph and T is an Euler tour in $H(t)$. Each visit to vertex v in T involves entry and exit through distinct edges, resulting in an increase of 2 in the part degree, $d(v)[H(t)]$. Any subsequent occurrences of v in T also add 2 each to $d(v)[H(t)]$, ensuring it remains an even number. Likewise, given that T initiates and concludes at u, the initial and final edges each contribute 2 to $d(u)[H(t)]$, and any other instances of u within T contribute 2 each to $d(u)[H(t)]$. Consequently, $d(u)[H(t)]$ is also an even number. This means that $d(v)[H(t)]$ is even for all v in $H(t)$. As this was proven for an arbitrarily selected part $H(t)$, it follows that this holds true for every part $H(t)$ of F.

Conversely assume that $F = (F^*, R, S, T)$ be a soft graph of F with all parts $H(t)$ connected and $d(v)[H(t)]$ is even for all v in $H(t)$ and in every part $H(t)$ containing v. Let $H(t)$ be a part of F for some $t \in T$. To demonstrate that $H(t)$ is an Eulerian part, we employ mathematical induction on the number of edges $|S(t)|$ of $H(t)$, which involves starting with a base case and proving that the proposition holds true for that case. We then show that if the proposition is true for a particular case, it must also hold true for the next case. If $|S(t)| =0$, i.e., if $H(t)$ has no edges then $H(t)$ contains only one vertex u since $H(t)$ is a connected part. Then $T = u$ is an Euler tour in $H(t)$ and so $H(t)$ is an Eulerian part of F. Now assume that $H(t)$ has more edges. Since $H(t)$ is connected $d(v)[H(t)]$ must be at least two for every vertex v in $H(t)$. So, by the above theorem $H(t)$ contains a cycle C. If C contains all edges in the part $H(t)$, C will be an Euler tour in $H(t)$. If not, remove the edges contained in C from $H(t)$. Then, we get a new subgraph $J(t)$ of $H(t)$ having less edges than $H(t)$. $J(t)$ may be a disconnected graph and part degree of every vertex in $J(t)$ are still even. So, by our induction assumption all components of $J(t)$ contain an Euler tour. Additionally, because these components are created by removing edges that are part of C, each component will share a vertex with C. We can then generate an Euler tour in $H(t)$ by following these steps: start at any vertex in C and continue until reaching a common vertex, which we'll call v_1, with one of the components in $J(t)$. Then, traverse an Euler tour within that component before returning to v_1. Next, travel along the edges of C until reaching another common vertex, v_2, with a different component in $J(t)$. After completing an Euler tour of that new component, return to v_2. Continue this process and come back to the beginning vertex after visiting the Euler tours in all components of $J(t)$. This produces an Euler tour in the part $H(t)$ and hence $H(t)$ is Eulerian. Since $H(t)$ is arbitrary we can say that every part of F is an Eulerian part. Hence, F *is* an Eulerian soft graph.

Corollary 3.1. Let $F = (F^*, R, S, T)$ be an Eulerian soft graph of the graph $F^* = (X, D)$ and $W = \bigcup_{t \in T} R(t)$ be the vertex set of F. Then for any vertex v in W, $d_F(v)$ is even and hence $D(F)$ is even.

Proof. Let $F = (F^*, R, S, T)$ be an Eulerian soft graph. Then by Theorem 3.3, the part degree $d(v)[H(t)]$ of a vertex v is even in every part $H(t)$ of F containing v. If v is not present in a part $H(t)$, then we treat $d(v)[H(t)]$ as zero. Hence for any vertex v in W, $d_F(v) = \Sigma_{t \in T}\, d(v)$ [H(t)] is even and hence $D(F) = \Sigma_{v \in W} d_F(v)$ is even.

Theorem 3.4. Let $F = (F^*, R, S, T)$ be a soft graph of $F^* = (X, D)$ which is represented by $\{H(t): t \in T\}$. Let $H(t)$ be any part of F that is connected. Then $H(t)$ contains an Euler trail if and only if there exist at most two vertices in $H(t)$ having an odd part degree.

Proof. Assume that the connected part $H(t)$ of F contains an Euler trail T starting at u and ending at v. Then by the proof of the first part of theorem 3.2, part degree of every vertex of $H(t)$ other than u and v is even. The only possible vertices in $H(t)$ having odd part degrees are u and v.

Conversely, assume that there are at most two vertices having odd part degrees in the connected part $H(t)$ of the soft graph F. If $H(t)$ has no vertex having odd part degree then $d(v)[H(t)]$ is even for every vertex v of $H(t)$. Then, by theorem 3.3, $H(t)$ is an Eulerian part. So, it contains an Euler tour and hence an Euler trial. Assume that $H(t)$ has precisely two vertices with an odd part degree, denoted as u and v. We can then examine $H(t) + uv$, which is formed by adding the edge uv to $H(t)$. Then uv increases the part degree of u and v by 1 and hence part degree of every vertex of $H(t) + uv$ is even. Therefore, $H(t) + uv$ will be Eulerian and will contain an Euler tour C containing the edge uv. Deleting the edge uv from C we get an Euler trail in $H(t)$.

Theorem 3.5. Consider $F = (F^*, R, S, T)$, a soft graph derived from $F^* = (X, D)$, where every part is a connected component. Then F *is* Eulerian if and only if each part $H(t)$ of F is the union of edge-disjoint cycles.

Proof. Suppose that the soft graph F is Eulerian. Then its part $H(t)$ is Eulerian, $\forall t \in T$. Consider any such Eulerian part $H(t)$. We use mathematical induction on $|S(t)|$ where $|S(t)|$ denotes the count of edges in the part $H(t)$ to prove that $H(t)$ is the union of edge-disjoint cycles. If $|S(t)| = 3$, then $H(t)$ is the complete graph K_3 and is a cycle. Assume that the result is true when $|S(t)| < m$ where $m > 3$. Let $|S(t)|$ be m. Since $H(t)$ is an Eulerian part, $d(v)[H(t)]$ is even for every vertex v of $H(t)$. Thus $H(t)$ is a connected graph which is not a tree. So, $H(t)$ contains at least one cycle, say C. If C is a cycle on m vertices, then $H(t) = C$ and the result is proved. If not, there are edges in $H(t)$ which are not in C. Remove all edges of C from $H(t)$ to produce a graph $Q(t)$ in which part degree of every vertex is even. Now applying the induction assumption to all nontrivial components of $Q(t)$ we can say that they are the union of edge disjoint cycles. Now all these cycles together with cycle C give the required result. Since $H(t)$ is arbitrary we can say that each part $H(t)$ of the Eulerian soft graph F is the union of edge-disjoint cycles.

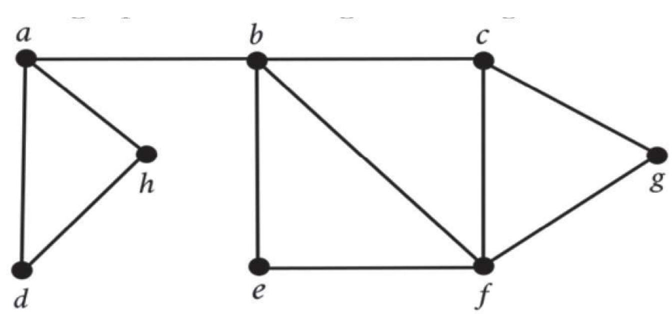
Conversely, assume that each part $H(t)$ of the soft graph F is the union of edge-disjoint cycles. Then the part degree of every vertex in $H(t)$ is even for every t of T So, by theorem 3.3, $H(t)$ is an Eulerian part for every $t \in T$, and hence F is an Eulerian soft graph.

4. Unicursal Soft Graphs

Definition 4.1. Let $F = (F^*, R, S, T)$ represent a soft graph of F^* defined by $\{H(t): t \in T\}$. In this context, any part $H(t)$ of F is termed unicursal if it contains a unicursal line. In other words, $H(t)$ includes an open trail that traverses every edge within $H(t)$, or $H(t)$ features an open Euler trail.

Definition 4.2. A soft graph $F = (F^*, R, S, T)$ derived from a graph F^* is referred to as unicursal if every part $H(t)$ of F is unicursal.

Example 4.1. Consider the graph $F^* = (X, D)$ given in *Fig. 9*.



Let $T = \{a, f\}$ be a parameter set. Define a mapping $R: T \rightarrow P(X)$ by $R(t)=\{y \in X | tRy \Leftrightarrow d(t,y) \leq 1\}$ for all $t \in T$.

That is, R(a) = {a, b, d, h} and R(f) = {b, c, e, f, g}.

Also define another mapping S: T→P (D) by S(t) = $\{uv \in D | \{u,v\} \subseteq R(t)\}$ for all t ∈T. That is, *S(a) = {ab, ah, ad, dh} and S(f) = {bc, be, bf, cf, cg, ef, fg }*. Here (R, T) and (S, T) are soft sets over X and D respectively. Thus H(a) = (R(a), S(a)) and H(f) = (R(f), S(f)) are subgraphs of F* and is shown in Fig. 10. So, F= {H(a), H(f)} is a soft graph of F*.

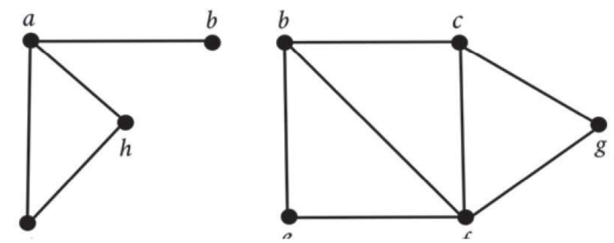


Fig. 10. F = {*H(a), H(f)*}

Table 5 provides the tabulated depiction of this soft graph.

T/X	*a*	*b*	*c*	*d*	*e*	*f*	*g*	*h*
a	1	1	0	1	0	0	0	1
f	0	1	1	0	1	1	1	0

T/D	*ab*	*ad*	*ah*	*bc*	*be*	*bf*	*cf*	*cg*	*dh*	*ef*	*fg*
a	1	1	1	0	0	0	0	0	1	0	0
f	0	0	0	1	1	1	1	1	0	1	1

Table 5

Here *F* has *H(a)* and *H(f)* as parts. In *H(a)*, T_1 = *adhab* is a unicursal line and so *H(a)* is a unicursal part. In *H(f)*, T_2 = *befbcfgc* is a unicursal line and so *H(f)* is a unicursal part. That is, all parts of *F* are unicursal and so *F is* a unicursal soft graph.

Theorem 4.1. Consider *F = (F*, R, S, T)*, a soft graph derived from *F* = (X, D)*, and represented by {*H(t): t*∈*T*}. Assume that each part *H(t)* of *F* is connected. Then *F* is a unicursal soft graph if and only if there exist exactly two vertices each in all parts *H(t)* having an odd part degree.

Proof. Let *F* be a unicursal soft graph and the connected part *H(t)* of *F* contains a unicursal line *T* starting at *u* and ending at *v*. Join *u* and *v* by an edge to get a new graph $H'(t)$. This graph $H'(t)$ will be Eulerian and hence the degree of every vertex in $H'(t)$ will be even. Removal of this edge reduces the degrees of both *u* and *v* by one. Hence the only two vertices in *H(t)* having odd part degrees are *u* and *v* and all other vertices have even part degrees. This is true for all parts *H(t)* of *F*.

Conversely, assume that there are exactly two vertices each having odd part degrees in the connected parts *H(t)* of the soft graph *F* for all *t* in *T*. Let *H*(*t*) be any such part and *u* and *v* be two vertices with odd part degrees. We can then examine *H*(*t*) + *uv,* which is formed by adding the edge *uv* to *H*(*t*). Then *uv* increases the part degree of *u* and *v* by 1 and hence degrees of all vertices of *H(t)* + *uv* is even. Therefore, *H(t)* +*uv* will be Eulerian and will contain an Euler tour *C* containing the edge *uv*. Deleting the edge *uv* from *C* we get a unicursal line in *H(t).* This is true in all parts *H(t)* of *F*. That is, all parts *H(t)* of *F* are unicursal and hence *F* is a unicursal soft graph.

5. Conclusion

Soft set theory offers a versatile framework for addressing uncertain and imprecise information, aspects not comprehensively addressed by classical set theory. The introduction of soft sets gave rise to the concept of soft graphs, allowing for the generation of diverse representations of a relation represented by a graph through parameterization. The capability of soft graphs to accommodate parameterization has propelled the theory of soft graphs into a rapidly advancing domain within graph theory. In this article, the authors presented the concept of Eulerian and unicursal soft graphs and explored some of their characteristics.

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