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Lyapunov Stability of Tethered Dumbbell Satellites in Elliptical Orbit

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Abstract

This paper represents the equilibrium positions and stability of two artificial satellites connected by light, flexible, and elastic long tethers under the combined effect of several classical perturbative forces in an elliptical orbit. The tether may be conducting or non-conducting. In our problem, it is taken as non-conducting in nature. We have treated the problem by taking five perturbative forces on the system simultaneously. Three perturbations exist due to the influences of the earth, namely geomagnetic fields, shadows, and oblateness. The other perturbations are due to the elasticity of the cable and solar light pressure. The effect of air resistance is neglected, considering the satellites as high-altitude satellites. To determine the stability of the satellites, we have used the Lyapunov method. The dynamical behaviors of the satellites are represented by differential equations. Based on analytical analysis of the differential equations of motion, we get the equilibrium positions of the system concerned in elliptical orbit. Lyapunov method gives the equilibrium position as unstable as expected.

Keywords: Tether, Satellites, Shadow, Oblateness, Geomagnetic field, Solar light pressure

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1. Introduction

Over the past few decades, tether connected satellite systems have attracted the attention of researchers as an important and challenging space technology [1-5]. It has an extremely wide range of applications and a variety of problems depending upon the choice of model and perturbative forces. The exploration of space, various transport operations, generation of electricity, creation of artificial gravity, and many other tasks can be solved with the help of tether connected satellite systems. The characteristics of tether connected satellites are that they have long lengths, variable configurations, and can interact with the geomagnetic field. The presence of various perturbative forces makes the dynamical behaviours of the systems very complicated. For many application purposes, stationary movement of the tether is most suitable. In many studies of tether satellites, the researcher assumed the satellite's system moves in a circular orbit and obtained valuable remarks. A detailed investigation of the stationary motion of tethers and their stability was conducted by Beletsky and Levin [6]. They showed that in a circular orbit, the motion of the systems and their elastic vibrations are unstable. In most of the studies, stability problems are carried out in the absence of other generalised perturbing forces. Burov and Troger studied the relative equilibriums and stability conditions for the tether satellite system [7]. The process of solving the absolute stability of a dynamical system is studied by Liberzon [8]. Yu et al. [9] reviewed the dynamics, modeling, and stability of tethered satellite systems. Kumar and Kumar [10] studied the equilibrium positions of a cable-connected satellite system under several influences. Yu et al. [11] also studied the chaotic motion of tethered satellite systems under the effect of air drag and Earth's oblateness.

In this paper, we are interested in calculating the equilibrium positions and stability of the systems in Keplerian elliptical orbit under various perturbative forces mentioned in the abstract, taking the dumbbell model of the satellite system. Although the magnitude of Earth's magnetic field and solar radiation pressure is very small, the effects of these perturbative forces are very important because the system is exposed to these perturbative forces over a long time. We have taken the tether as non-conducting and elastic in nature. The shadow of the earth is taken as cylindrical in nature, and electrostatic interaction due to charges developed in the metal body of the satellite system during its motion is neglected for its small value. Lyapunov test is applied to the equilibrium position to determine the stability of the systems.

2. Modelling and Equilibrium Position

Modelling is very important to gain insight into the Dynamics of tether connected satellite systems. The three models that are very popular for tether satellites are the continuous model, discrete model, and rod model. Generally, the dynamical equations of these three models are constructed using Lagrange's equations, Hamilton's principle, and Newton's laws. The equations obtained under different perturbative forces of the tether satellites are generally nonlinear and non-autonomous. In our problem, we used the rod model of tether satellites for the analytical analysis. Consider a satellite system connected by a non-conducting and elastic tether with the center of mass moving along a Keplerian elliptical orbit. We have already derived [12, 13] the dynamical equations in a rotating coordinate system in the body-centred frame of reference for such a model, and is written as

$$X'' - 2Y' - \frac{3X}{\left(1-e^2\right)^{1/2}} = -A\cos i + \frac{R^3}{\mu}\left(\frac{B_1}{m_1} - \frac{B_2}{m_2}\right)\frac{\cos\in\cos\alpha\sin\theta}{\pi} + \frac{12k_2}{R^2}\frac{1}{\left(1-e^2\right)^{1/2}}X - \lambda_\alpha\frac{1}{\left(1-e^2\right)^{1/2}}\frac{R^3}{\mu}\left[1 - l_0\left(X^2+Y^2\right)^{-1/2}\right]X$$

$$Y'' + 2X' = \frac{R^3}{\mu}\left(\frac{B_1}{m_1} - \frac{B_2}{m_2}\right)\frac{\cos\in\sin\alpha\sin\theta}{\pi} - \frac{3k_2}{R^2}\frac{1}{\left(1-e^2\right)^{1/2}}Y - \lambda_\alpha\frac{R^3}{\mu}\frac{1}{\left(1-e^2\right)^{1/2}}\left[1 - l_0\left(X^2+Y^2\right)^{-1/2}\right]Y \tag{1}$$

Where, $\lambda_\alpha = \left[\dfrac{m_1+m_2}{m_1 m_2}\right]\dfrac{\lambda}{l_0}$, $k_2 = \dfrac{\bar{\varepsilon}R_e^{\,2}}{3}$, $\bar{\varepsilon} = \alpha_R - \dfrac{m}{2}$, $m = \dfrac{\Omega^2 R_e}{g_e}$

$$A = \left(\frac{m_1}{m_1+m_2}\right)\left(\frac{Q_1}{m_1} - \frac{Q_2}{m_2}\right)\frac{\mu_E}{\sqrt{\mu p}}$$

Where m_1 and m_2 are the masses of the mother satellite and sub-satellite. μ denotes the earth's gravity parameter. λ denotes the elastic parameter for the connecting cable. Q_1, Q_2 are the charges developed on the two satellites respectively. B_1, B_2 are the absolute values of the forces due to the direct solar radiation pressure on m_1 and m_2, respectively. l_0 is the length of the tether, α_R denotes the earth's oblateness, Ω denotes the angular velocity on the earth's rotation, R_e is for the equatorial radius of the earth, and g_e is the acceleration due to gravity. e is the eccentricity of the orbit, i is the inclination of the orbit with the equatorial plane, $\in$ is the inclination of the oscillatory plane of the masses m_1 and m_2 with the orbital planes of the centre of the mass of the system, and α is the inclination of the ray θ is called the shadow angle. The magnetic moment of the earth's dipole is represented by μ_e and p is the focal parameter. The prime represents differentiation with respect to the true anomaly (v).

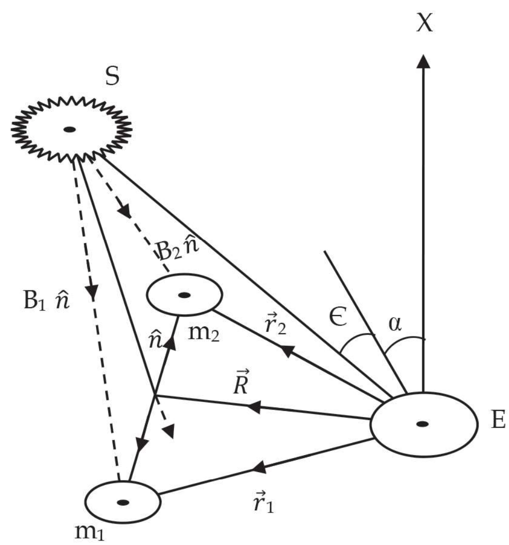

Fig. 1: Diagrammatical representation of the cable-connected satellites system under several influences

Lagrangian points are given by the constant values of the coordinate in the rotating frame of reference. Let these points be represented by

$$X=X_0 \quad \text{and} \quad Y=Y_0 \tag{2}$$

When the system's center of mass is located at a certain point in an elliptical or circular orbit, the equilibrium state is chosen as constant values of coordinates with respect to the system's body-centered coordinate. Physically, it means that the speed and acceleration of the system are zero with respect to its body-centred coordinate system. Therefore, the first and second derivative of the coordinates of equation (2) is zero.

Putting (2) in the set of equations (1) we obtain

$$-\frac{3X_0}{\left(1-e^2\right)^{1/2}}=-A\cos i+\frac{R^3}{\mu}\left(\frac{B_1}{m_1}-\frac{B_2}{m_2}\right)\frac{\cos\in\cos\alpha\sin\theta}{\pi}+\frac{12k_2}{R^2}\frac{1}{\left(1-e^2\right)^{1/2}}X_0-\lambda_\alpha\frac{1}{\left(1-e^2\right)^{1/2}}\frac{R^3}{\mu}\left[1-l_0\left(X^2+Y^2\right)^{-1/2}\right]X_0$$

$$\frac{R^3}{\mu}\left(\frac{B_1}{m_1}-\frac{B_2}{m_2}\right)\frac{\cos\in\sin\alpha\sin\theta}{\pi}=\frac{3k_2}{R^2}\frac{1}{\left(1-e^2\right)^{1/2}}\mathrm{Y}_0+\lambda_\alpha\frac{R^3}{\mu}\frac{1}{\left(1-e^2\right)^{1/2}}\left[1-l_0\left(X^2+Y^2\right)^{-1/2}\right]\mathrm{Y}_0 \tag{3}$$

It is very difficult to obtain the analytical solution of equation (3). Here, we are compelled to make our approaches with suitable limitations. Apart from this, we are interested only in obtaining the maximum effect of the earth's shadow on the motion of the system.

In the further investigation, we put $\in=0$ and $\alpha=0$ as because $\left(\frac{B_1}{m_1}-\frac{B_2}{m_2}\right)$ or θ cannot be zero. Therefore, we shall write (3) as

$$-\frac{3X_0}{\left(1-e^2\right)^{1/2}}=-A\cos i+\frac{R^3}{\mu}\left(\frac{B_1}{m_1}-\frac{B_2}{m_2}\right)\frac{\sin\theta}{\pi}+\frac{12k_2}{R^2}\frac{1}{\left(1-e^2\right)^{1/2}}X_0-\lambda_\alpha\frac{1}{\left(1-e^2\right)^{1/2}}\frac{R^3}{\mu}\left[1-l_0\left(X^2+Y^2\right)^{-1/2}\right]X_0$$

$$\frac{3k_2}{R^2}\frac{1}{\left(1-e^2\right)^{1/2}}\mathrm{Y}_0=-\lambda_\alpha\frac{R^3}{\mu}\frac{1}{\left(1-e^2\right)^{1/2}}\left[1-l_0\left(X^2+Y^2\right)^{-1/2}\right]\mathrm{Y}_0 \tag{4}$$

Where $r_0=\left(\mathrm{X}_0{}^2+\mathrm{Y}_0{}^2\right)^{1/2}$

These two equations are independent of each other. From the first equation of (4), it follows X- coordinate of the equilibrium point cannot be zero as A and θ are non-vanishing. The system is wholly extended along the X-axis. In this case, $\mathrm{Y}_0=0$. Thus we shall have the equilibrium point as $(\mathrm{X}_0, 0)$

Next, from the first equation of (4), we have

$$-\frac{3X_0}{\left(1-e^2\right)^{1/2}}=-A\cos i+\frac{R^3}{\mu}\left(\frac{B_1}{m_1}-\frac{B_2}{m_2}\right)\frac{\sin\theta}{\pi}+\frac{12k_2}{R^2}\frac{1}{\left(1-e^2\right)^{1/2}}X_0-\lambda_\alpha\frac{1}{\left(1-e^2\right)^{1/2}}\frac{R^3}{\mu}\left[1-l_0\left(X^2+Y^2\right)^{-1/2}\right]X_0$$

or,

$$\frac{3X_0}{\left(1-e^2\right)^{1/2}}+\frac{12k_2}{R^2}\frac{1}{\left(1-e^2\right)^{1/2}}X_0-\lambda_\alpha\frac{1}{\left(1-e^2\right)^{1/2}}\frac{R^3}{\mu}X_0=A\cos i-\lambda_\alpha\frac{1}{\left(1-e^2\right)^{1/2}}\frac{R^3}{\mu}l_0-\frac{R^3}{\mu}\left(\frac{B_1}{m_1}-\frac{B_2}{m_2}\right)\frac{\sin\theta}{\pi}$$

$$X_0=\frac{\left(1-e^2\right)^{1/2}\left[A\cos i-\lambda_\alpha\frac{1}{\left(1-e^2\right)^{1/2}}\frac{R^3}{\mu}l_0-\frac{R^3}{\mu}\left(\frac{B_1}{m_1}-\frac{B_2}{m_2}\right)\frac{\sin\theta}{\pi}\right]}{(3+\frac{12k_2}{R^2}-\lambda_\alpha\frac{R^3}{\mu})} \tag{5}$$

Therefore, the equilibrium position of motion of the system concerned is written as

$$(X_0,\mathrm{Y}_0)=\frac{\left(1-e^2\right)^{1/2}\left[A\cos i-\frac{\lambda_\alpha R^3}{\mu\left(1-e^2\right)^{1/2}}l_0-\frac{R^3}{\mu}\left(\frac{B_1}{m_1}-\frac{B_2}{m_2}\right)\frac{\sin\theta}{\pi}\right]}{(3+\frac{12k_2}{R^2}-\lambda_\alpha\frac{R^3}{\mu})},0 \tag{6}$$

From equation (6), we can easily determine the equilibrium position in a circular orbit for the same problem by just putting the eccentricity value zero, so for the circular orbit

$$(X_0,\mathrm{Y}_0)=\left(\frac{\left[A\cos i-\lambda_\alpha\frac{R^3}{\mu}l_0-\frac{R^3}{\mu}\left(\frac{B_1}{m_1}-\frac{B_2}{m_2}\right)\frac{\sin\theta}{\pi}\right]}{(3+\frac{12k_2}{R^2}-\lambda_\alpha\frac{R^3}{\mu})},0\right) \tag{7}$$

3. Stability of the Equilibrium Position

To test the stability of the equilibrium position of the tether satellite system under the mentioned perturbative forces, we will rewrite the equations (1) in the case of the maximum effect of earth's shadow and putting $\in=0$ and $\alpha=0$.

$$X''-2Y'-\frac{3X}{\left(1-e^2\right)^{1/2}}=-A\cos i+\frac{R^3}{\mu}\left(\frac{B_1}{m_1}-\frac{B_2}{m_2}\right)\frac{\sin\theta}{\pi}+\frac{12k_2}{R^2}\frac{1}{\left(1-e^2\right)^{1/2}}X-\lambda_\alpha\frac{1}{\left(1-e^2\right)^{1/2}}\frac{R^3}{\mu}\left[1-l_0\left(X^2+Y^2\right)^{-1/2}\right]X$$

$$Y''+2X'=-\frac{3k_2}{R^2}\frac{1}{\left(1-e^2\right)^{1/2}}Y-\lambda_\alpha\frac{R^3}{\mu}\frac{1}{\left(1-e^2\right)^{1/2}}\left[1-l_0\left(X^2+Y^2\right)^{-1/2}\right]Y \tag{8}$$

Now, we will take small variations in the coordinates of the equilibrium position and apply these variations to equations (8). Let the variations be represented by

$$X=X_0+\Delta_1$$
$$Y=Y_0+\Delta_2 \tag{9}$$

Applying these variations to equations (8), the couple of equations become

$$\Delta_1''-2\Delta_2'-\frac{3(X_0+\Delta_1)}{\left(1-e^2\right)^{1/2}}=-A\cos i+\frac{R^3}{\mu}\left(\frac{B_1}{m_1}-\frac{B_2}{m_2}\right)\frac{\sin\theta}{\pi}+\frac{12k_2}{R^2}\frac{(X_0+\Delta_1)}{\left(1-e^2\right)^{1/2}}-\frac{\lambda_\alpha R^3}{\mu\left(1-e^2\right)^{1/2}}\left[1-l_0[(X_0+\Delta_1)^2+\Delta_2^2]^{-1/2}\right](X_0+\Delta_1)$$

$$\Delta_2''+2\Delta_1'=-\frac{3k_2}{R^2}\frac{\Delta_2}{\left(1-e^2\right)^{1/2}}-\frac{\lambda_\alpha R^3}{\mu\left(1-e^2\right)^{1/2}}\left[1-l_0[(X_0+\Delta_1)^2+\Delta_2^2]^{-1/2}\right]\Delta_2 \tag{10}$$

Equations (8) admit the Jacobean integral, so its variation equations (10) also constitute the Jacobean integral. The form of the Jacobean integral in the variation parameters takes the form as

$$\Delta_1'^2+\Delta_2'^2-\left[\frac{2\lambda_\alpha R^3}{\mu\left(1-e^2\right)^{1/2}}l_0-\frac{2\lambda_\alpha R^3}{\mu\left(1-e^2\right)^{1/2}}X_0+\frac{6X_0}{\left(1-e^2\right)^{1/2}}-2A\cos i+\frac{2R^3}{\mu}\left(\frac{B_1}{m_1}-\frac{B_2}{m_2}\right)\frac{\sin\theta}{\pi}+\frac{24k_2}{R^2}\frac{X_0}{\left(1-e^2\right)^{1/2}}\right]\Delta_1+$$
$$\left[\frac{3k_2}{R^2\left(1-e^2\right)^{1/2}}+\frac{\lambda_\alpha R^3}{\mu\left(1-e^2\right)^{1/2}}\right]\Delta_2^2+\left[\frac{3}{\left(1-e^2\right)^{1/2}}-\frac{12K_2}{R^2\left(1-e^2\right)^{1/2}}+\frac{\lambda_\alpha R^3}{\mu\left(1-e^2\right)^{1/2}}\right]\Delta_1^2=h \tag{11}$$

Where h is the Jacobean constant. To test the stability, we will now apply the Lyapunov method to the Jacobean integral equation. This integral equation is considered as Lyapunov function $L(\Delta_1',\Delta_2',\Delta_1,\Delta_2)$. Thus

$$L(\Delta_1',\Delta_2',\Delta_1,\Delta_2)=\Delta_1'^2+\Delta_2'^2-\left[\frac{2\lambda_\alpha R^3}{\mu\left(1-e^2\right)^{1/2}}l_0-\frac{2\lambda_\alpha R^3}{\mu\left(1-e^2\right)^{1/2}}X_0+\frac{6X_0}{\left(1-e^2\right)^{1/2}}-2A\cos i+\frac{2R^3}{\mu}\left(\frac{B_1}{m_1}-\frac{B_2}{m_2}\right)\frac{\sin\theta}{\pi}+\frac{24k_2}{R^2}\frac{X_0}{\left(1-e^2\right)^{1/2}}\right]\Delta_1$$
$$+\left[\frac{3k_2}{R^2\left(1-e^2\right)^{1/2}}+\frac{\lambda_\alpha R^3}{\mu\left(1-e^2\right)^{1/2}}\right]\Delta_2^2+\left[\frac{3}{\left(1-e^2\right)^{1/2}}-\frac{12K_2}{R^2\left(1-e^2\right)^{1/2}}+\frac{\lambda_\alpha R^3}{\mu\left(1-e^2\right)^{1/2}}\right]\Delta_1^2 \tag{12}$$

The Lyapunov function $L(\Delta_1',\Delta_2',\Delta_1,\Delta_2)$ is the integral of the variations equations (10); its differentiation taken along the trajectory of the system must vanish identically. The only condition for this is that the Lyapunov function must be positive definite. For making equation (12) positive definite, the first-order variable terms should be zero. In fact, these conditions imply equilibrium positions. The second-order terms must satisfy Sylvester's conditions for positive definiteness. So, the sufficient condition becomes

(i) $$\frac{2\lambda_\alpha R^3}{\mu\left(1-e^2\right)^{1/2}}l_0-\frac{2\lambda_\alpha R^3}{\mu\left(1-e^2\right)^{1/2}}X_0+\frac{6X_0}{\left(1-e^2\right)^{1/2}}-2A\cos i+\frac{2R^3}{\mu}\left(\frac{B_1}{m_1}-\frac{B_2}{m_2}\right)\frac{\sin\theta}{\pi}+\frac{24k_2}{R^2}\frac{X_0}{\left(1-e^2\right)^{1/2}}=0$$

(ii) $$\frac{3}{\left(1-e^2\right)^{1/2}}-\frac{12K_2}{R^2\left(1-e^2\right)^{1/2}}+\frac{\lambda_\alpha R^3}{\mu\left(1-e^2\right)^{1/2}}>0$$

(iii) $$\frac{3k_2}{R^2\left(1-e^2\right)^{1/2}}+\frac{\lambda_\alpha R^3}{\mu\left(1-e^2\right)^{1/2}}>0$$

4. Conclusions

The aim of the present paper is to obtain the equilibrium position and stability of the tether satellites under the influence of several perturbative forces like a shadow of the earth, solar radiation pressure, oblateness of the earth, and earth's magnetic field in elliptical orbit when the tether is elastic in nature. We have obtained the equilibrium position under these conditions. The equilibrium condition shows that the Y centroid orbit coordinate is zero. This indicates the tether coincides with the plane of orbit with the center of mass, and the mass points of m_1 and m_2 are collinear with the center of mass of the earth. From equations (6) and (7), it is clear that the X coordinated is more extended when the center of mass moves in a circular orbit compared to an elliptical orbit, and also, it is widest for a circular orbit. However this equilibrium position is not stable because the sufficient conditions (ii & iii) for the stability are not satisfied simultaneously. The perturbations influencing the system make the system unstable. For a small period of time, we may neglect the other perturbative forces, and then we may have stable equilibrium positions. To stabilize the orbit, we need different control methods, which are the other parts of the tether satellite research.

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