Mapana – Journal of Sciences
2024, Vol. 23, No. 1, 35-45
ISSN 0975-3303 | https://doi.org/10.12723/mjs.68.2

Features of Projectile Motion in Quantum Calculus

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Abstract

In this paper, the equation of motion of a projectile in a resistive medium is revisited in view of quantum calculus. Quantum calculus, abbreviated as q-calculus is a recently recognized unconventional type of calculus in which q-derivatives of a real function can be obtained without limit. In order to describe the motion of a 2-D projectile, vertical and horizontal components of velocity in terms of q-differential equations are presented in accordance with the exact classical expressions. The solutions are obtained in terms of the small q-exponential function. Features of a projectile are also deducted one by one for relevance. It is also found that, when the parameter $q \to 1$, the present solutions and all the features take the form as they are in general Newtonian classical mechanics.

Keywords: projectile, quantum calculus, q-derivative, q-exponential, 2-D projectile

1. Introduction

The Leibniz notation df(x)/dx is the ratio of two infinitesimal uses limits in calculating the derivative of a real function whereas calculus without limits is known as quantum calculus. Up until today Leibnitz notation is widely used in every branch in science unambiguously. But a subsequent development in q-derivative and q-integral can also be useful to analyze physical systems in a different way as mentioned and described in Ref [1, 2, and 3]. In addition, q-calculus shows a

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different interesting way to find out exact solutions to several models in statistical mechanics, heat and wave equations [4-8]. Recently, the falling body problem, the very basic problem in Newtonian classical mechanics is also described in view of q-calculus [9]. Here, the projectile problem is analyzed extensively, based on the basics of q-calculus as described in ref. [1-9]

Projectile motion is the simplest type of 2-D motion in Newtonian classical mechanics that can be studied easily by means of any different type of calculus [8-12]. In this paper, the motion of a 2-D projectile is studied by using basic formulae of q-calculus. To describe the dynamical equations, at first, the problem is systematically analysed by considering the two different components of velocity of the particle. It is also shown that the corresponding solutions to X and Y component of velocity, range, height and equation of motion will be reduced to the classical Newtonian ones when the elementary parameter 'q' tends to 1.

Essential Preliminaries of Quantum Calculus

(I) Here $[m]_q=\frac{q^m-1}{q-1}$ for any positive integer m such that $\lim_{q\to1}[m]_q=m$ (introductory chapter in Ref. **[1]).** Thus, $[0]_q=0$ and $[1]_q=1$.

(II) If f(t) is an arbitrary function, its q-differential (as in chapter 1 of Ref. **[1]**) $d_qf(t)=f(qt)-f(t)$ is and the q- derivative is $D_qf(t)=\frac{d_qf(t)}{d_qt}=\frac{f(qt)-f(t)}{(q-1)t}$. It is also to be noted that $\lim_{q\to1} D_qf(t)=\frac{df(t)}{dt}$

Thus, $D_qt^m=\frac{(qt)^m-(t)^m}{(q-1)t}=\frac{q^m-1}{q-1}t^{\,m-1}=[m]_qt^{\,m-1}$.

(III) *q-integral*: The Jackson q-integral (definite) is given as (chapter 19 of Ref. **[1]**) $\int_0^a f(t)d_qt=(1-q)a\sum_{l=o}^{\infty} q^lf(q^la)$, Hence $\int_0^a D_qf(t)d_qt=f(a)-f(0)$

(IV) *q-factorial:* The quantum factorial $[m]_q!$ is defined as (in 3rd chapter of Ref. [1])

$$[m]_q!=\begin{matrix}[0]_q!=1\\ \prod_{m=1}^{m}[m]_q\end{matrix}$$

(v) *q-exponential*: $e_q(t)$ is basically the small q-exponential function of t is represented as (described in chapter 9 of Ref. **[1]**)

$$e_q(Rt)=\sum_{l=0}^{\infty}\frac{(Rt)^l}{[m]_q!}$$

(vi) *And* $\int e_q(Rt)d_q t=\frac{1}{R}e_q(Rt)+c$ where *c* is a real constant.

3. The Problem

Suppose initial velocity of the projected particle of mass *M* is *u* and θ is initial angle of projection with the horizontal base. Therefore, at t=0 $u_x=u\cos\theta$ and $u_y=u\sin\theta$ are horizontal and vertical components of velocity respectively. The Newtonian equations of motion for such a particle moving in a resistive medium are given by,

$$M\frac{du_x}{dt}=-MRu_x \text{ or } \frac{du_x}{dt}=-Ru_x \tag{1}$$

$$\frac{du_y}{dt}=-g-Ru_y \tag{2}$$

Where, R is a real positive constant of dimension $[T^{-1}]$ and represents the strength of the resistive force. The initial conditions are

$$u_x(\text{at } t=0, x=0)=u\cos\theta,\ u_y(\text{at } t=0, x=0)=u\sin\theta \tag{3}$$

The classical solution of (1) and (2) are

$$u_x(t)=(u\cos\theta)\exp(-Rt) \tag{4}$$

$$u_y(t)=-(g/R)+(g/R+u\sin\theta)\exp(-Rt) \tag{5}$$

Integrating (4) and (5) with proper initial condition and rearranging, we can arrive at,

$$x(t)=\frac{u\cos\theta}{R}[1-\exp(-Rt)] \tag{6}$$

$$y(t)=-(g/R)t+\frac{(g/R+u\sin\theta)}{R}\left(1-\exp(-Rt)\right) \tag{7}$$

Eliminating 't', from (6) and (7), the equation of motion is obtained as

$$y(x)=+(g/R^2)\ln(1-xR/u\cos\theta)+(g/R+u\sin\theta)\,(x/u\cos\theta) \tag{8}$$

Now, 'time of flight (T)', 'Maximum Height '(Y_M)', and 'horizontal range 'X_M' are

$$T=\frac{2}{R}\ln\left[1+\frac{u\sin\theta}{g}R\right] \tag{9}$$

$$Y_{max}=-(g/R)\frac{T}{2}+\frac{(g/R+u\sin\theta)}{R}\left[1-\exp\left(-R\frac{T}{2}\right)\right] \tag{10}$$

And,

$$X_{max}=\frac{u\cos\theta}{R}[1-\exp(-RT)] \tag{11}$$

4. Application of Quantum Calculus to the Problem

In view of q-calculus, (1) and (2) can be written as,

$$\frac{d_q u_x}{d_q t}=-Ru_x \tag{12}$$

$$\frac{d_q u_y}{d_q t}=-g-Ru_y \tag{13}$$

In order to solve these two equations, we assume the solutions as

$$u_x(t)=\sum_{m=0}^{\infty}A_m t^m \tag{14}$$

And

$$u_y(t)=\sum_{m=0}^{\infty}B_m t^m \tag{15}$$

Where, A_m and B_m are constants to be determined from proper initial conditions.

Thus,

$$\frac{d_q u_x(t)}{d_q t}=\sum_{m=0}^{\infty}[m]_q A_m t^{m-1} \text{ and } \frac{d_q u_y(t)}{d_q t}=\sum_{m=0}^{\infty}[m]_q B_m t^{m-1}$$

Horizontal Motion:

Considering the definition of $[m]_q$, $[0]_q=0$, in order to solve the x-component of velocity,

$$\frac{d_q u_x(t)}{d_q t} = \sum_{m=1}^{\infty}[m]_q A_m t^{m-1} = \sum_{m=0}^{\infty}[m+1]_q A_{m+1} t^m \tag{16}$$

$$\frac{d_q u_y(t)}{d_q t} = \sum_{m=1}^{\infty}[m]_q B_m t^{m-1} = \sum_{m=0}^{\infty}[m+1]_q B_{m+1} t^m \tag{17}$$

Substituting, (16) in (12), we get

$$\sum_{m=0}^{\infty}[m+1]_q A_{m+1} t^m = -R\sum_{m=0}^{\infty} A_m t^m$$

Hence,

$$A_1[1]_q + \sum_{m=1}^{\infty}[m+1]_q A_{m+1} t^m = -RA_0 - R\sum_{m=1}^{\infty} A_m t^m$$

Then, comparing the coefficient of t^0 and t^m,

$$A_1 = \frac{-RA_0}{[1]_q}$$

$$A_{m+1} = \frac{-RA_m}{[m+1]_q} \qquad \text{for } m \geq 1 \tag{18}$$

From (18)

$$A_2 = \frac{-RA_1}{[2]_q} = \frac{(-R)^2 A_0}{[1]_q[2]_q}$$

$$A_3 = \frac{-RA_2}{[3]_q} = \frac{(-R)^3 A_0}{[1]_q[2]_q[3]_q}$$

And hence,

$$A_m = \frac{(-R)^m A_0}{[1]_q [2]_q [3]_q \cdots [m]_q} = \frac{(-R)^m A_0}{[m]_q!}$$

Thus,

$$u_x(t) = \sum_{m=0}^{\infty} A_m t^m = A_0 + \sum_{m=1}^{\infty} A_m t^m$$

$$= A_0 + A_0 \sum_{m=1}^{\infty} \frac{(-Rt)^m}{[m]_q!}$$

Now, considering both the boundary condition at t = 0, $u_x(0)$ = $ucos\theta$,

$$u_x(t) = u\cos\theta \left[1 + \sum_{m=1}^{\infty} \frac{(-Rt)^m}{[m]_q!}\right]$$

Introducing the concept of small q-exponential (as mentioned in section 2), $e_q\left(-Rt\right) = \sum_{m=0}^{\infty} (-Rt)^m / [m]_q!$, above equation becomes,

$$u_x(t) = u\cos\theta[1 + \{e_q(-Rt) - 1\}]$$

Therefore, the instantaneous X-component of velocity is given by,

$$u_x(t) = (u\cos\theta) e_q(-Rt) \tag{19}$$

In order to calculate, the arbitrary horizontal distance x(t), we use the concept of q-integration as

$$\int_0^t D_q x(t') d_q t' = \int_0^t (u\cos\theta) e_q(-Rt') d_q t' \tag{20}$$

Where, $u_x(t) = D_q x(t)$. Integrating (20) with the use of q-exponential concept, we obtain,

$$[x(t')]_0^t = \left(-\frac{u\cos\theta}{R}\right)\left[e_q(-Rt')\right]_0^t \tag{21}$$

Or,

$$[x(t)-x(0)]=\left(\frac{u\cos\theta}{R}\right)\left[1-e_q(-Rt)\right] \tag{22}$$

As initially,at t=0, the particle was at x=0, (22) gives the horizontal position at *t* as

$$x(t)=\left(\frac{u\cos\theta}{R}\right)\left[1-e_q(-Rt)\right] \tag{23}$$

Vertical Motion:

Similarly, for the *y*-component of velocity, substituting (17) in (13) we obtain

$$\sum_{m=0}^{\infty}[m+1]_q B_{m+1}t^m=-g-R\sum_{m=0}^{\infty}B_m t^m \tag{24}$$

Then,

$$B_1[1]_q+\sum_{m=1}^{\infty}[m+1]_q A_{m+1}t^m=-g-RB_0-R\sum_{m=1}^{\infty}B_m t^m$$

Comparing the coefficient of t^0 and t^m,

$$B_1=\frac{-g-RB_0}{[1]_q}$$

$$B_{m+1}=\frac{-RB_m}{[m+1]_q}\ \text{for } m\geq 1 \tag{25}$$

From (25)

$$B_2=\frac{-RB_1}{[2]_q}=\frac{(-1)^2Rg+(-R)^2B_0}{[1]_q[2]_q}$$

$$B_3=\frac{-RB_2}{[2]_q}=\frac{(-1)^3R^2g+(-R)^3B_0}{[1]_q[2]_q[3]_q}$$

And hence,

$$B_m = \frac{(-1)^m R^{m-1} g + (-R)^m B_0}{[1]_q [2]_q [3]_q \cdots [m]_q} t^m$$

$$= \frac{(-1)^m R^{m-1} g + (-R)^m B_0}{[m]_q!} t^m$$

Thus,

$$u_y(t) = \sum_{m=0}^{\infty} B_m t^m = B_0 + \sum_{m=1}^{\infty} B_m t^m$$

$$= B_0 + \sum_{m=1}^{\infty} \left[\frac{(-Rt)^m \left(g/R\right) + (-Rt)^m B_0}{[m]_q!} \right]$$

Now, considering both the boundary condition at $t = 0, u_y(0) = usin\theta = B_0$

$$u_y(t) = usin\theta + \left(g/R + usin\theta\right) \sum_{m=1}^{\infty} \frac{(-Rt)^m}{[m]_q!}$$

Introducing the concept of q -exponential (as mentioned in section 2), $e_q\left(-Rt\right) = \sum_{m=0}^{\infty} (-Rt)^m / [m]_q!$, above equation becomes,

$$u_y(t) = usin\theta + (g/R + usin\theta)\ (e_q(-Rt) - 1)$$

Therefore, at an arbitrary moment t, Y-component of velocity is given by,

$$u_y(t) = -(g/R) + (g/R + usin\theta)(e_q(-Rt) \tag{26}$$

In order to calculate the arbitrary vertical distance $y(t)$, we use the concept of q-integration as

$$\int_0^t D_q y(t') d_q t' = \int_0^t -(g/R) d_q t' + \int_0^t \left(g/R + usin\theta\right) e_q(-Rt') d_q t' \tag{27}$$

Where, $u_y(t) = D_q y(t)$. Integrating (27) with small q-exponential concept, we obtain,

$$[y(t')]_0^t = -\left(g/R\right) \left[\frac{t'}{[1]_q} \right]_0^t + \frac{\left(g/R + usin\theta\right)}{R} \left[-e_q(-Rt')\right]_0^t \tag{28}$$

Or,

$$[y(t)-y(0)]= -(g/R)\left(\frac{t}{[1]_q}\right)+\frac{(g/R+u\sin\theta)}{R}\left(1-e_q(-Rt)\right) \tag{29}$$

As initially, at t=0, the particle was at y=0, and $[1]_q=1$. Hence, (29) gives the vertical position at *t* as

$$y(t)= -(g/R)t+\frac{(g/R+u\sin\theta)}{R}\left(1-e_q(-Rt)\right) \tag{30}$$

Solving (23), we have

$$t=-\frac{1}{R}\log_q\left(1-\frac{xR}{u\cos\theta}\right) \tag{31}$$

Now, substituting the value of 't' as obtained in (31) into (30), we obtain

$$y(x)=(g/R^2)\log_q\left(1-\frac{xR}{u\cos\theta}\right)+(g/R+u\sin\theta)(x/u\cos\theta) \tag{32}$$

5. Important Features of Projectile and Dimensionality

Time of flight:

If time of flight is T, then considering the symmetry of the projectile trajectory the particle reaches at the maximum height at T/2 and $u_y(T/2)=0$, Hence from (26)

$$-(g/R)+(g/R+u\sin\theta)e_q(-RT/2)=0$$

$$T=\frac{2}{R}\log_q\left(1+\frac{u\sin\theta}{g}R\right) \tag{33}$$

Maximum Height and horizontal Range:

Substituting T, in (30) and (23) respectively, the maximum height and Range of the projectile are given by

$$Y_{max}= y\left(\frac{T}{2}\right)=-(g/R)\frac{T}{2}+\frac{(g/R+u\sin\theta)}{R}\left[1-e_q\left(-R\frac{T}{2}\right)\right] \tag{34}$$

$$X_{max}=\left(\frac{u\cos\theta}{R}\right)\left[1-e_q\left(-R\frac{T}{2}\right)\right] \tag{35}$$

Here, it is important to be noted that (19) & (26) should exactly be reduced to the corresponding classical one if and only if $q\rightarrow1$. In addition (30) to (35) will reduce to the exact classical one if both $q\rightarrow 1$ and $\log_q e_q^x = x$. The dimensionality of all the parameters also depend on the dimension of e_q^{Rt} and $(1\text{-}e_q^{Rt})$. As the dimension of R is $[T^{-1}]$, Rt, e_q^{Rt}, $(1\text{-}e_q^{Rt})$ are dimensionless quantity, there is no issue regarding the dimensionality of any parameters.

6. Comparative Analysis

It is shown that (19), (23), (26), (30) and (32) to (35) reduces to its exact classical form as displayed from (4) to (11). Here we investigate that as $q\rightarrow 1$, $\lim_{q\rightarrow 1} \ln_q x = \ln x$. In addition, the equation of motion of a projectile in q-calculus will be reduced to its classical form if and only if the relation between small exponential and q-logarithm are related as $\log_q e_q = 1$. Thus properties of q-logarithm and q-exponential in Tsallis statistics is also successfully holds to be true for projectile motion

7. Conclusion

In this paper, a projectile of point mass under a drag force proportional to the instantaneous velocity is studied in view of quantum calculus. Several more ways to find the equation of motion of a throwing particle has been investigated but the approach discussed in the present paper is completely new one. For the first time it is investigated for projectile motion that the general condition $q\rightarrow 1$ transforms the q-solution to its corresponding classical one. As a generalization, the proposed relation between q-exponential and q-logarithm is verified in case of projectile motion without any ambiguity. It is also to be noted that the dimensional analysis is also discussed to investigate the correctness of the dimensionality of all the deducted formulae. The present work can further be extended to deduce the equation of SHM and many other oscillatory systems in classical physics.

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