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Dirac Particle in Crossed Electric and Magnetic Field

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Abstract

Electromagnetic forces play a critical role in shaping the trajectories of charged particles, affecting their movement across scales ranging from subatomic dimensions to distances smaller than astronomical ones. This study focuses on the behaviour of a charged spin-half particle exposed to a perpendicular electric and magnetic field. We analyze the associated eigenfunctions and eigenvalues using the Landau gauge.The non-relativistic and classical limits align with the standard results.

Keywords: Bound States, Crossed Electric and Magnetic Fields, Landau Levels, Dirac Equation, Landau Gauge.

1. Introduction

A Charged particle moving in a region with both electric and magnetic fields will experience the Lorentz force, which causes them to follow a curved trajectory [1]. Depending on the initial

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velocity, the trajectory can be a trochoid or a cycloid [2]. In the presence of an electric field having a component perpendicular to the magnetic field, the motion is a combination of drift and spiral motion aligned along the direction of the magnetic field and the direction of drift is independent of the charge of the particle[3] [4].

In the presence of a strong electric field ($E > Bc$), the charged particle will be accelerated along the direction of the electric field and the motion is unbounded. However, in the presence of a strong magnetic field ($E < Bc$), the charged particle will experience a looping motion around the magnetic field direction and a drifting motion in a direction perpendicular to both the applied fields, making the motion semi bounded.

The relativistic quantum states for such a system have been very well discussed [5], [6], [7].

In this article, the motion of a charged Dirac particle in crossed-electric and magnetic fields is discussed quantum mechanically to obtain the exact quantized states and the results are compared with the classical results.

2. Solutions to the Dirac Equation

Consider a spin half particle with charge q and mass m placed in a set of uniform crossed electric and magnetic fields. Let the electric field be along y axis and magnetic field be along the z axis. Therefore, the electrostatic potential and vector potential are respectively, given by

$$\Phi = -qEy \tag{1}$$

$$\vec{A} = -By\hat{i} \tag{2}$$

Where we are using the Landau gauge. Since the motion occurs in x–y plane, we may ignore the motion along the z direction and hence the corresponding momentum along the z direction (Pz) is set to zero.

Now the Dirac Hamiltonian in the standard form is given by

$$H = c(\alpha_x p_x + \alpha_y p_y + \alpha_z p_z) + \beta mc^2 \quad (3)$$

where c is the speed of light and $\alpha_i, \alpha_y, \alpha_z$ and β are the Dirac matrices which can be written as

$$\alpha_i = \begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix} \quad (4)$$

i=1,2,3 with σ_i being the two-dimensional Pauli spin matrices. Then,

$$\beta = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \quad (5)$$

For the magnetic field along the z direction, we can choose

$$H = c(\sigma_x p_x + \sigma_y p_y) + \sigma_z mc^2 \quad (6)$$

for the motion in the $x - y$ plane.

Explicitly, $\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\sigma_y = \begin{bmatrix} 0 & -i \\ i & o \end{bmatrix}$ and $\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Using the minimal coupling rule we can write the equation for the two component wave function ψ as

$$(H - \Phi)\Psi = c\left[\sigma_x(p_x - qA_x) + \sigma_y(p_y - qA_y) + mc\sigma_z\right]\Psi \quad (7)$$

Putting in the potentials we get

$$(H + qEy)\Psi = c\left[\sigma_x(p_x + qBy) + \sigma_y p_y + mc\sigma_z\right]\Psi \quad (8)$$

in other words,

$$\begin{bmatrix} \epsilon + qEy & 0 \\ 0 & \epsilon + qEy \end{bmatrix}\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} mc^2 & c(P_x + qBy) - icP_y \\ c(P_x + qBy) + icP_y & -mc^2 \end{bmatrix}\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \quad (9)$$

where ψ_1 and ψ_2 are the two-components of ψ and ε is the eigenvalue of H. The equations satisfied by ψ_1 and ψ_2 are:

$$\left(\epsilon + qEy - mc^2\right)\psi_1 = \left[c(p_x + qBy - icP_y)\right]\psi_2 \tag{10a}$$

$$\left(\epsilon + qEy + mc^2\right)\psi_2 = \left[c(p_x + qBy + icP_y)\right]\psi_1 \tag{10b}$$

the "small" component ψ_2 can be written as

$$\psi_2 = \frac{c(p_x + qBy + icP_y)}{\epsilon + qEy + mc^2}\psi_1 \tag{11}$$

substituting (11) in (10a) we get for the "large component" ψ_1

$$\left(\epsilon + qEy - mc^2\right)\psi_1 = \left[c(p_x + qBy - icP_y)\right] \times \frac{c(p_x + qBy + icP_y)}{\epsilon + qEy + mc^2}\psi_1 \tag{12}$$

simplifying (12) we get

$$\left((\epsilon + qEy)^2 - m^2c^4\right)\psi_1 = c^2\left\{(p_x + qyB)^2 + P_y^2 + i(P_x + qyB)P_y - iP_y(Px + qBy)\right\}\psi_1 \tag{13}$$

Using the commutation relations of angular momentum, (13) reduces to

$$\left((\epsilon + qEy)^2 - m^2c^4\right)\psi_1 = c^2\left[P_x^2 + q^2y^2B^2 + 2qyP_xB + P_y^2 - qB\hbar\right]\psi_1 \tag{14}$$

In a similar wayan eigenvalue equation for ψ_2 can be obtained Clearly, the above equation does not contain terms of x therefore P_x will be a constant.

Choosing the eigenvalue of $P_x = a$, a constant, we get

$$\frac{1}{c^2}\left[\epsilon^2 + q^2E^2y^2 + 2\epsilon Ey - m^2c^4\right]\psi_1 = \left[P_y^2 + 2qB\alpha y + q^2B^2y^2 + \alpha^2 - qb\hbar\right]\psi_1 \tag{15}$$

This equation will lead to a bound state if $\left(q^2B^2 - q^2\frac{E^2}{c^2}\right) > 0$. Rewriting (15) gives

$$P_y^2\psi_1 + \omega^2\left(y^2 + \frac{2q(\alpha B - \frac{\epsilon}{c^2}E)}{\omega^2}y\right)\psi_1 = \bar{\epsilon}\psi_1 \tag{16}$$

where

$$\omega^2 = q^2B^2 - q^2\frac{E^2}{c^2} \quad \text{the strong magnetic field case} \tag{17}$$

and

$$\bar{\epsilon} = \frac{1}{c^2}(\epsilon^2 - m^2c^4) - \alpha^2 + qB\hbar \tag{18}$$

changing the variable to $\bar{y} = \left(y + \frac{q(\alpha B - \frac{\epsilon}{c^2}E)}{\omega^2}\right)$, equation (16) can be written as

$$(P_{\bar{y}}^2 + \omega^2\bar{y}^2)\psi_1 = \left[\bar{\epsilon} + \frac{q^2(\alpha B - \frac{\epsilon}{c^2}E)^2}{\omega^2}\right]\psi_1. \tag{19}$$

This is nothing but the equation for a simple hormonic oscillator moving around $\bar{y} = 0$. The eigenvalues of this system can be immediately written as

$$2\left(n + \frac{1}{2}\right)\hbar\omega = \bar{\epsilon} + \frac{q^2(\alpha B - \frac{\epsilon}{c^2}E)^2}{\omega^2} \quad ; \quad n = 0, 1, 2, ... \tag{20}$$

Solving for ε, using the condition for a strong magnetic field, we get

$$\epsilon \approx \left[(mc^2)^2 + (\alpha c)^2 - qB\hbar c^2 + 2\left(n + \tfrac{1}{2}\right)\hbar c^2\left(q^2B^2 - q^2\tfrac{E^2}{c^2}\right)\right]^{\frac{1}{2}} \tag{21}$$

It can also be written as

$$\epsilon \approx mc^2\left[1 + \tfrac{(\alpha c)^2}{2m^2c^4} - \tfrac{\mu.B}{mc^2} + \tfrac{1}{2m^2c^4}\left(n + \tfrac{1}{2}\right)\hbar c^2\left(q^2B^2 - q^2\tfrac{E^2}{c^2}\right)\right]^{\frac{1}{2}} \tag{22}$$

where $\mu = \frac{q\hbar}{2m}$, is the magnetic dipole moment of the particle. It can be observed that (23) contains the rest energy and the kinetic energy corresponding to the motion along the x direction. The energy due to oscillatory motion is seen in the last term with the frequency $\omega = \sqrt{\left(B^2 - \frac{E^2}{c^2}\right)}$. It is also evident the Landau levels are reproduced in $E = 0$. It is noteworthy to see that the third term corresponds to the potential energy of magnetic dipole moment in the applied magnetic field. The eigenfunctions corresponding to the eigenvalues (19) are given by

$$\psi_1 = \left(\frac{\frac{\omega}{\hbar}}{\sqrt{\pi}2^n n!}\right)^{\frac{1}{2}} exp\left(-\frac{\omega^2}{2\hbar^2}\bar{y}^2\right) H_n\left(\frac{\omega\bar{y}}{\hbar}\right) \tag{23}$$

Where $H_n\left(\frac{\omega\bar{y}}{\hbar}\right)$ are the Hermite polynomials in $\bar{y}$ and of the order n. To account for the continuous linear uniform motion in the x direction, a factor of $exp\left(\frac{i\alpha x}{\hbar}\right)$ has to be multiplied with ψ_1.

3. Results and Discussions

The paper presents the study of a charged spin-half particle in crossed electric and magnetic fields and yields rich insights into the interplay of external forces and quantum behaviour. we establish that a Dirac particle's motion in the y-direction is restricted, resembling the drift motion observed in spin-zero particles moving along the x-direction.

It highlights the importance of the particle's magnetic dipole moment in shaping its potential energy and dynamics. Interestingly, under the condition $E = 0$, the eigenvalues seamlessly transition into the well-known Landau levels which align with non-relativistic limits, affirming our theoretical framework's validity.

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