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A Characterization of the Total Graph of Interval Graphs and Proper Interval Graphs

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Abstract

An asteroidal triple is a stable set of three vertices such that each pair is connected by a path avoiding the neighbourhood of the third vertex. Let G be a graph with vertex set V (G) and edge set E(G), the total graph T(G) of G has vertex set V (G) U E(G) and two vertices in T(G) are adjacent if and only if they are adjacent or incident in G. In this paper, we try to characterize the total graph of interval graphs and proper interval graphs.

Keywords: Asteroidal triple, interval graph, proper interval graphs, total graph, triangular graph

Mathematics Subject Classification: 05C75

1. Introduction

Graph colouring has become a subject of great interest because of its diverse theoretical results and its numerous applications. The total colouring was introduced by M. Behzad [2]. In 1960s M. Behzad and Vizing independently conjectured that $\chi''(G) \leq \Delta(G) + 2$. Where $\chi''(G)$ is the total chromatic number. In 1965 M. Behzad introduced a new graph called total graph. The total graph of a simple graph $G = (V, E)$ is a graph for which V (T(G)) = V U E and such that two distinct vertices in T(G) are adjacent if and only if they are adjacent vertices of G or adjacent edges of G or they are incident vertex and edge of G. Now, the total colouring of G can be simply viewed as the vertex colouring of T(G).

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In 1970, M. Behzad [3] characterized total graphs in terms of special points. In 1969 M. Behzad and Heydar Radjavi [4] gave the structure of regular total graphs. In 2020 T.B. Athul and G. Suresh Singh [1] gave some characterization of total graphs of regular graphs. In this paper we try to characterize the total graph of interval graphs. A graph G is said to be an interval graph if its vertices have a one-to-one correspondence with a collection of intervals in the real line such that two vertices are adjacent in G if and only if their corresponding intervals have nonempty intersection. Throughout this paper we consider simple, connected and finite graphs.

A graph G is said to be a triangular graph if every simple cycle of length greater than four possesses a chord.

Theorem 1.1. [7] Every interval graph satisfies triangulated property.

An asteroidal triple in a graph G is a set of three non-adjacent vertices such that for any two of them, there exists a path between them in G that does not intersect the neighbourhood of the third. The graph in Figure 1 is an example of the smallest graph that contains an asteroidal triple; the three vertices forming the asteroidal triple are encircled.

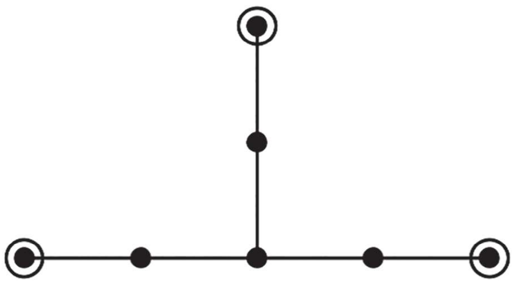


Figure 1: Smallest graph contains an asteroidal triple

Theorem 1.2. [5] A triangular graph is an interval graph if and only if it contains no asteroidal triple.

A proper interval graph is an interval graph in which no interval properly contains the other intervals. In [6], Zygmunt Jackowski characterized the proper interval graphs in terms of astral triple. Three vertices *x y, z* in a graph *G* form an astral triple if between any vertices there exists a path P in G such that the third vertex of the triple does not belong to P and any two subsequent vertices in P are not both adjacent to the third vertex of the triple. Also, from [6] it is clear that, every asteroidal triple is an astral triple.

Theorem 1.3. [6] G is a proper interval graph if and only if G contains no astral triple.

Theorem 1.4. [6] The following conditions are equivalent:

i. G is proper interval graph

ii. G is chordal and does not contain the following graphs as an induced subgraph.

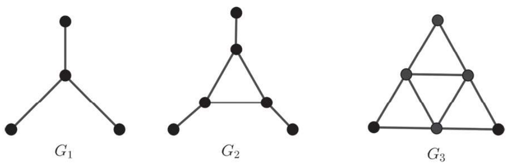


Figure 2

Throughout this paper we consider simple finite graphs.

2. Total Graphs and Interval Graphs

In this section we give some condition for which $T(G)$ is triangular and interval graphs. Further, we characterize total graph of triangular and interval graphs.

Theorem 2.1. *A graph G has triangulated property if and only if T(G) has triangulated property.*

Proof. Suppose $T(G)$ has triangulated. Since G is an induced subgraph of G, so G has triangulated property.

Conversely let us G has triangulated property. Clearly $L(G)$ has triangulated property. Suppose there exists a cordless cycle $v_1, v_2, \ldots, v_n$ in $T(G)$, for $n > 3$. Then these vertices can be partitioned in to two, if necessary relabel the vertices, as $V_1 = (v_1, v_2, \ldots, v_p)$ and $V_2 = (v_{p+1}, v_{p+2}, \ldots, v_n)$ corresponding to the vertices and edges of G. Since v_{p+1} is incident with v_p in G, let $v_{p+1} = v_p u_1$, for some $u_1 \in V(G)$. Since v_{p+1} and

v_{p+2} are adjacent in *T(G)*, the corresponding edges in *G* are adjacent. If v_{p+2} is adjacent with v_p then the vertex v_p, v_{p+1} and v_{p+2} forms a triangle in *T(G)*, which is not possible, hence v_{p+2} is adjacent with u_1, let $v_{p+2} = u_1u_2$, since v_{p+2} is adjacent with v_{p+3} in *T(G)*, so v_{p+3} is incident with u_2, let $v_{p+3} = u_2u_3$. Proceeding like this, finally we get $v_n = u_{n-(p+1)} u_{n-p}$. Since v_n is incident on v_1, u_{n-p} must be v_1. Now we get a cycle $v_1, v_2, \ldots, v_p, u_1, u_2, \ldots, v_1$ in G. It is clear that v_i and v_{i+2} are not adjacent in G. Suppose u_1 and u_3 are adjacent in *G*, then consider the cycle $v_1, v_2, \ldots, v_p, u_1, u_2, u_4, \ldots v_1$ in G. Proceeding like this we get a cycle in *G* with no chords, which is a contradiction.

Remark. If *T(G)* is an interval graph then *G* is an interval graph. But the converse need not be true.

Example 2.2. Consider the following graph *G* with line graph *L(G)*. The circled vertices form an asteroidal triples so, *T(G)* is not an interval graph.

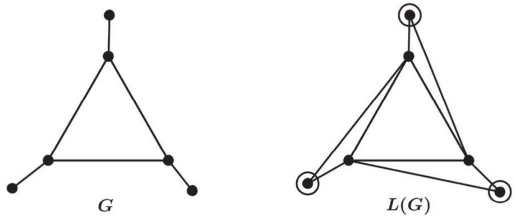


Figure 3: A graph *G* with *L(G)*

Next, we try to bring a characterization for a total graph *T(G)* to be an interval graph.

Theorem 2.3. *T(G)* is an interval graph if and only if both *G* and *L(G)* are interval graphs.

Proof. First assume that *T(G)* is an interval graph. Then clearly *G* and *L(G)* are interval graphs.

Conversely suppose that *G* and *L(G)* are interval graphs, by Theorem 2.1, *T(G)* has triangulated property. Suppose the vertices v_1, v_2 and v_3 forms an asteroidal triple in *T(G)*. Clearly at least two of them is either in *V (G)* or in *V (L(G))*.

Case 1: Assume $v_1, v_2 \in V(G)$ and $v_3 \in V(L(G))$.

Let $v_3 = u_1u_2$, where u_1 and u_2 belongs to $V(G)$. Clearly v_1 and v_2 are not adjacent with u_1 and u_2. Let P_1 be the path joining v_1 and v_3 in $T(G)$ which doesn't contain a neighbour of v_2. If a vertex $v_4 = u_3u_4$ of $L(G)$ is in P_1, then u_3 and u_4 are not adjacent to v_2. Hence the path P_1 induces a path connecting v_1 and u_1 in G, which does not contain a neighbour of v_2. In the similar way we can find a path in G connecting the vertices v_1 and v_2 which does not contain any neighbour of u_1 and v_2 and u_1 which does not contain a neighbour of v_1. This follows that the vertices v_1, v_2, u_1 forms an asteroidal in G, which is a contradiction.

Case 2: If $v_1, v_2 \in V(L(G))$ and $v_3 \in V(G)$.

Let $v_1 = u_1u_2$ and $v_2 = u_3u_4$, where $u_i \in V(G)$ for $i = 1, 2, 3, 4$. Clearly the vertices u_i for $i = 1, 2, 3, 4$ are not adjacent to v_3. Let P_1 be a path in $T(G)$ joining v_1 and v_3 which does not contain a neighbour of v_2. If $e = uv$ be a vertex in P_1, then u_3 is not adjacent with u and v, hence P_1 induces a path joining u_1 and v_3 in G which does not contain a neighbour of u_3. Similarly, we can find $u_3 - u_1$ path in G which does not contain a neighbour of v_3 and $u_3 - v_3$ path in G which does not contain a neighbour of u_1. Hence the vertices u_1, u_3, v_3 form an asteroidal triple in G, which is a contradiction.

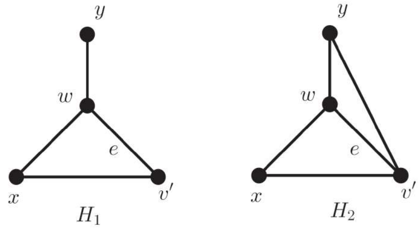
Hence by Theorem 1.2, $T(G)$ is an interval graph.

3. Characterization of Total Graphs in Terms of Proper Interval Graphs

In this section we try to study the relation between proper interval graphs and its total graphs.

The graph G_1 given in Theorem 1.4 is not a proper interval graph but its line graph $L(G)$ is C_3, which is a proper interval graph. In general the star graph $K_{1,n}$ is not a proper interval graph but $T(K_{1,n})$ is a proper interval graph. Hence, if G is proper interval graph then it does not imply that $L(G)$ is proper interval graph. Using this fact, we characterize the total graphs of proper interval graphs.

Theorem 3.1. *$T(G)$ is a proper interval graph if and only if both G and $L(G)$* are proper interval graphs and G does not contain H_1, H_2 as induced subgraphs.



Proof. First assume *T(G)* is a proper interval graph. Since *G* and *L(G)* are induced subgraphs of *T(G)*, both are proper interval graphs. If *G* contains H_1, H_2 as induced subgraphs then the vertices *x, y, w, e* in *T(G)* forms G_1 in Theorem 1.4 as an induced subgraph, which contradicts our assumption that *T(G)* is a proper interval graph.

Conversely assume that *G* does not contain H_1, H_2 as induced subgraphs and *G*, *L(G)* are proper interval graphs then by Theorem 1.2 and Theorem 2.3, *T(G)* does not contain asteroidal triple. It is clear that the graphs G_2 and G_3 given in Theorem 1.4 contains asteroidal triples. Hence it remains to prove that *T(G)* does not contain the graph G_1 in Theorem 1.4 as an induced subgraph.

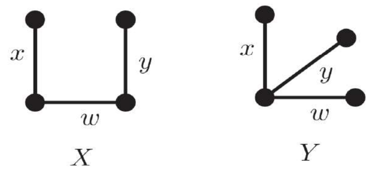
Suppose that *T(G)* contains G_1 as an induced subgraph. Let *w, x, y, z* be its vertices. Since *G* and *L(G)* are proper interval graphs, all these vertices do not belong to *V (G)* or *V (L(G))*. Then the following three cases may arise.

Case 1: Three vertices belong to *V (G)* and one vertex belongs to *V ((L(G))*.

If x, y, w $\in V(G)$, $z \in V(L(G))$ and $z = wv_1$ for some $v_1 \in V(G)$. If v_1 is not adjacent to *x* or *y* then, *x, y, w, v′* forms an astral triple in *G*, which is not possible. Hence v' must be adjacent to *x* or *y* in each case the vertex *x, y, w, v* induced subgraph isomorphic to either H_1 or H_2 in *G*, is a contradiction.

Case 2: Three vertices belong to *V ((L(G))* and one in *V (G)*.

If $x, y, w \in V(L(G))$ and $z \in V(G)$ and *z* is adjacent to *w* in *T(G)*, then $w = zv_1$ in *G*, where $v_1 \in V(G)$. The subgraph induced by the vertices *w, x, y* in *L(G)* is one of the following graphs.



Since *L(G)* is proper interval graph the existence of *y* is not possible. If *X* as an induced subgraph then *z* is either adjacent to *x* or *y* in *T(G)*, which is a contradiction.

Case 3: Two vertices belong to *V (G)* and two in *V (L(G))*.

If $x, y \in V(G), w, z \in L(V(G))$ then *x* and *y* are either adjacent to *w* or *z*. Without loss of generality, assume that *x* and *y* are adjacent to *w*, then $xy \in E(G)$, is not possible.

Hence, from the three cases, it is clear that G_1 is not an induced subgraph of *T(G)*, which completes the proof.

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