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Separation Axioms Associated with Simple Digraphs and Topological Spaces

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Abstract

The main idea of this article is to define fuzzy crisp set, intuitionistic crisp set and neutrosophic crisp set from simple digraphs. These sets have its own impact to generate the subbasis and which in turn yields topological spaces. Moreover, an attempt has been made to extend our concept in induced subgraphs that leads us to relative topology. We have also formalized the structural equivalence of the isomorphic graphs and the topologies induced by them. A comparison between topologies has been made for some types of connected digraphs. Also, we have defined separation axioms on digraphs and related them to the topological separation axioms.

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1. Introduction

Konigsberg bridge problem is a milestone in graph theory which was the back bone to many emerging research problems. Study of deformation of shapes under continuous transformation was the initiator of topology. In recent days topology which was considered as a pure analytic theory has also stepped in solving some real-life application problems. To enrich the essence of both, these two major theories were cloned as topological graph theory. Many researchers have innovated much more in topologizing the graph by its vertices,

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edges, adjacency and labelling. Among them AdemKilicman and Khalid Abdulkalek[1]in 2018 have assessed the topologies induced by digraphs.

The notion of fuzzy sets($\mathcal{F}$-sets) was introduced by L.A.Zadeh[2] (1965). In recent days many enactments behind uncertainty pertain to the concept of $\mathcal{F}$-sets and its logic. Later, as an extension, Atanassov[3](1986) explored the idea of $\mathfrak{I}\mathcal{F}$ sets and $\mathfrak{I}\mathcal{VF}$ sets. Further FloretinSmarandache developed this into neutrosophic sets and neutrosophic crisp sets($\mathcal{N}\mathbb{C}$-sets) in which indeterministic values are also taken into account.

In this article we generate topological spaces from simple directed graphs by defining $\mathcal{F}$c-set, $\mathfrak{I}$c-set and $\mathcal{N}$c-set for every vertex of the graph. This collection yields us the subbasis which in turn generates a topological space. We also studied that there is a homeomorphism between the topologies which are induced by the graphs that are isomorphic. Moreover, we extend this concept in induced subgraph which generate relative topology. Also, we have defined separation axioms on digraphs such as T_0-graph, T_1-graph and T_2-graph. In addition, we have also investigated some relations between digraph separation axioms and topological separation axioms.

2. Preliminaries

Few concepts of graphs and topology essential for our work are given in this section.

Definition 2.1[4]A directed graph (D-graph) $\vec{\mathcal{G}}$ consists of a set of nodes N= {n_1, n_2,…}, a set of Links L= {l_1, l_2,...} and a mapping ‘σ’ that maps every links onto some ordered pair of nodes (n_i, n_j).

Definition 2.2[4] D-graphs having at most one directed link between a pair of nodes, but are allowed to have self-loops are known as asymmetric D-graphs.

Definition 2.3[4] The number of links incident out of a node n_i is called the out-degree of n_i. The number of links incident into n_i is called the in-degree of n_i. If the in-degree and the out-degree of a node both equal to zero then it is called an isolated node.

Definition 2.4[5] A D-graph $\vec{\mathcal{G}}_1$is isomorphic to a D-graph $\vec{\mathcal{G}}_2$, if there is a function φ : N($\vec{\mathcal{G}}_1$) →N($\vec{\mathcal{G}}_2$) which is both 1-1 and onto, such that (u, v) ϵ L($\vec{\mathcal{G}}_1$) iff (φ(u), φ(v)) ϵ L($\vec{\mathcal{G}}_2$). Then, φ is known as an isomorphism from $\vec{\mathcal{G}}_1$ to $\vec{\mathcal{G}}_2$.

Definition 2.5[5] A D-graph $\vec{\mathcal{G}}$ is connected if the underlying graph G is connected. A D-graph $\vec{\mathcal{G}}$ is strong (strongly connected) if for every pair α, β of nodes, $\vec{\mathcal{G}}$ contains both α-β path and β-α path.

Definition 2.6[5] A connected D-graph is known as an Eulerian D-graph if it contains an Eulerian circuit.

Definition 2.7[5] A D-graph $\vec{\mathcal{G}}$ is Hamiltonian if it has a spanning cycle.

Definition 2.8[5] An oriented graph is a D-graph in which, if (α, β) is a link of $\vec{\mathcal{G}}$ then (β, α) is not a link of $\vec{\mathcal{G}}$. An oriented complete graph is known as a tournament.

Definition 2.9[6] Let (X, ℑ) be a topological space. If Y ⊆ X, then $\Im_Y$ = {Y ∩ U : U ϵ ℑ} is a topology on Y, called the relative topology(subspace topology).

Definition 2.10[6] Consider two topological spaces X and Y. Let f be a bijective map from X to Y. If both f and it's inverse are continuous, then f is called homeomorphism.

Definition 2.11[7]Let X, a non-void set, an $\mathfrak{I}\mathcal{F}$-set ζ is a set of the form ζ = (X, ζ_1, ζ_2), here ζ_1,ζ_2 ⊆ X satisfies $\zeta_1 \cap \zeta_2$ = Φ. The sets ζ_1 and ζ_2 are the set of members and non-members of ζ respectively.

Definition 2.12[8]Let X, a non-void set, a $\mathcal{N}\mathbb{C}$-set ξ is a set, that havea form ξ = (ξ_1, ξ_2, ξ_3), here ξ_1, ξ_2 and ξ_3 ⊆ X satisfies ξ_1, ξ_2 and ξ_3 are pairwise disjoint.

Throughout this paper $\mathcal{F}c\vec{\mathcal{G}}$ - set, $\mathfrak{I}c\vec{\mathcal{G}}$- set and $\mathcal{N}c\vec{\mathcal{G}}$- set means fuzzy crisp$\vec{\mathcal{G}}$-set, intuitionistic crisp$\vec{\mathcal{G}}$-set and neutrosophic crisp$\vec{\mathcal{G}}$-set respetively. Also, D-graph means simple asymmetric digraph.

3. Topology induced by fuzzy crisp$\vec{\mathcal{G}}$-Set ($\mathcal{F}c\vec{\mathcal{G}}$- set)

Definition 3.1 Let $\vec{\mathcal{G}}$ = (N, L) be a D-graph with O(v), I(v) ≠ Φ, ∀ v ϵ N($\vec{\mathcal{G}}$), where O(v) and I(v) denotes out-neighbours and in-neighbours of v respectively. Then, the fuzzy crisp set $\mathcal{F}c_v$ = <A>, where A is either a set of in-neighbours or out-neighbours of v.

Definition 3.2 Let $\vec{\mathcal{G}}$ = (N, L) be a D-graph with O(v), I(v) ≠ Φ, ∀ v ϵ N($\vec{\mathcal{G}}$). Then $\mathcal{S}_{\mathcal{F}c}$ = {G: G ϵ $\mathcal{F}c_v$: v ϵN($\vec{\mathcal{G}}$)} is a subbasis for a topology on N of $\vec{\mathcal{G}}$. Also, (N, $\Im_{\mathcal{F}c\vec{\mathcal{G}}}$) is called $\mathcal{F}c\vec{\mathcal{G}}$-topological space on N. A set U belongs to $\Im_{\mathcal{F}c\vec{\mathcal{G}}}$ is known as $\mathcal{F}c\vec{\mathcal{G}}$-open set and its complement is called $\mathcal{F}c\vec{\mathcal{G}}$-closed set.

Remark 3.3 The collection of fuzzy crisp sets form the subbasis, the topology generated from this is called $\mathcal{F}c\vec{\mathcal{G}}$-topological space.

Definition 3.4 If the collection of all subsets of N($\vec{\mathcal{G}}$) is $\mathcal{F}c\vec{\mathcal{G}}$-open then (N, $\Im_{\mathcal{F}c\vec{\mathcal{G}}}$) is called discrete $\mathcal{F}c\vec{\mathcal{G}}$-topological space on N.

Proposition 3.5 $\mathcal{S}_{\mathcal{F}c}$ is a subbasis for a topology on N of $\vec{\mathcal{G}}$.

Proof: Since every node in $\vec{\mathcal{G}}$ has in-neighbours and out-neighbours, U$\mathcal{F}c_v$ = N. Hence, all unions of finite intersections of elements of $\mathcal{S}_{\mathcal{F}c}$ forms a topology on N.

Example 3.6 Consider a D-graph $\vec{\mathcal{G}}$ = (N, L) with N = {ς, ε, φ, ϱ, ϑ}.

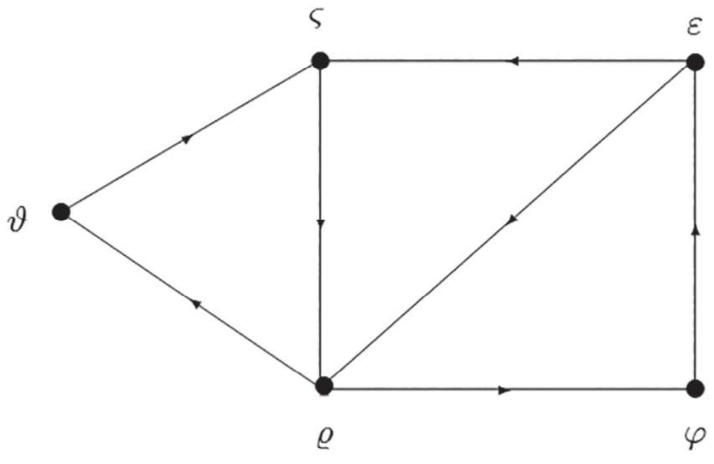

Figure 1. Graph for $\mathcal{F}c\vec{\mathcal{G}}$-topological space

Now, the $\mathcal{F}c_v$ sets for the vertices ς, ε, φ, ϱ, ϑ are as follows:

$\mathcal{F}c_ς$ = <{ε, ϑ}>; $\mathcal{F}c_ε$ = <{φ}>; $\mathcal{F}c_φ$ = <{ϱ}>; $\mathcal{F}c_ϱ$ = <{ς, ε}>; $\mathcal{F}c_ϑ$ = <{ϱ}>

So, the sub basis $\mathcal{S}_{\mathcal{F}c}$= {{ε, ϑ}, {φ}, {ϱ}, {ς, ε}}.

Hence $\Im_{\mathcal{F}c\vec{\mathcal{G}}}$ = {Φ, {ε}, {φ}, {ϱ}, {ς, ε} {ε, ϑ}, {ε, φ}, {ε, ϱ}, {φ, ϱ}, {ς, ε, φ},{ε, φ, ϑ}, { ς, ε, ϱ}, { ε, ϱ, ϑ}, { ε, φ, ϱ}, {ς, ε, ϑ}, {ς, ε, φ, ϱ}, {ς, ε, ϱ, ϑ}, {ε, φ, ϱ, ϑ}, N } is a $\mathcal{F}c\vec{\mathcal{G}}$- space on N.

Theorem 3.7 Every directed cycle graph $\overrightarrow{C_m}$, m≥ 3 generates a discrete $\mathcal{F}c\vec{\mathcal{G}}$-topology.

Proof: Let $\overrightarrow{C_m}$ = (N, L) be a directed cycle graph with N = {n_1, n_2, …, n_m}, L = {l_1, l_2, …, l_m}. Clearly each link l_i is an in-neighbour of n_{i+1} and an out-neighbour of n_i ∀i= 1,2,.., m. Hence $\mathcal{S}_{\mathcal{F}c}$= {{$n_1$}, {$n_2$}, …, {$n_m$}}. Clearly this subbasis generates a discrete $\mathcal{F}c\vec{\mathcal{G}}$-topology on N.

Theorem 3.8 Every vertex-induced subgraph $\vec{\mathcal{H}}$ of $\vec{\mathcal{G}}$ forms a relative topology $\Im_{\mathcal{F}c\vec{\mathcal{H}}}$.

Proof. Let $\vec{\mathcal{H}}$ = (N_1, L_1) be a vertex-induced subgraph of $\vec{\mathcal{G}}$ and $\Im_{\mathcal{F}c\vec{\mathcal{H}}}$ be a $\mathcal{F}c\vec{\mathcal{H}}$-topology on $N_1(\vec{\mathcal{H}})$. Then, for any two vertices u, v ϵ N_1, (u, v) ϵ L_1 if and only if (u, v) ϵ L. Let u ϵ N_1⊂N. Now, $\mathcal{F}c_{u_{\vec{\mathcal{H}}}}$={v : (u, v) ϵ$L_1$} and $\mathcal{F}c_{u_{\vec{\mathcal{G}}}}$ ={v : (u, v) ϵ L}. Clearly, $\mathcal{F}c_{u_{\vec{\mathcal{H}}}}$ = N_1 ∩ $\mathcal{F}c_{u_{\vec{\mathcal{G}}}}$. Here, $\mathcal{F}c_{u_{\vec{\mathcal{H}}}}$ is a subbasis element of $\Im_{\mathcal{F}c\vec{\mathcal{H}}}$. Hence, each element of subbasis of $\Im_{\mathcal{F}c\vec{\mathcal{H}}}$ is the intersection of N_1 and a subbasis element of $\Im_{\mathcal{F}c\vec{\mathcal{G}}}$. Let K be a-$\Im_{\mathcal{F}c\vec{\mathcal{H}}}$ open set. Clearly, K is the union of finite intersection of members of the subbasis $S_{\mathcal{F}c}$.

$$K = \cup_i\{\text{Basis elements of } \Im_{\mathcal{F}c\vec{\mathcal{H}}}\}$$

$$= \cup_i\{\cap_{i=1}^{n} \text{Subbasis elements of } \Im_{\mathcal{F}c\vec{\mathcal{H}}}\}$$

$$= \cup_i\left\{\cap_{i=1}^{n}(N_1 \cap \text{Subbasis elements of } \Im_{\mathcal{F}c\vec{\mathcal{G}}}\right\}$$

$$= \cup_i\left\{N_1 \cap (\cap_{i=1}^{n} \text{Subbasis elements of } \Im_{\mathcal{F}c\vec{\mathcal{G}}}\right\}$$

$$= N_1 \cap \cup_i\left\{\text{Basis elements of } \Im_{\mathcal{F}c\overrightarrow{\mathcal{G}}}\right\}$$

$= N_1 \cap$ R, where R is a $\Im_{\mathcal{F}c\vec{\mathcal{G}}}$ open set.

Therefore, $\Im_{\mathcal{F}c\vec{\mathcal{H}}}$ = {R ∩N_1 : R⊆$\Im_{\mathcal{F}c\vec{\mathcal{G}}}$}. Hence $\Im_{\mathcal{F}c\vec{\mathcal{H}}}$ is a relative topology of $\Im_{\mathcal{F}c\vec{\mathcal{G}}}$.

Theorem 3.9 If two D-graphs are isomorphic graphs then their corresponding $\mathcal{F}c\vec{\mathcal{G}}$-topologies are homeomorphic.

Proof. Let $\vec{\mathcal{G}}_1$ = (N', L') and $\vec{\mathcal{G}}_2$ = (N", L") be two simple digraphs with O(v), I(v) ≠ Φ for all nodes of $\vec{\mathcal{G}}_1$ and $\vec{\mathcal{G}}_2$. Let f be an isomorphism from $\vec{\mathcal{G}}_1$ to $\vec{\mathcal{G}}_2$. Let $\mathfrak{I}_{\mathcal{F}c\vec{\mathcal{G}_1}}$ and $\mathfrak{I}_{\mathcal{F}c\vec{\mathcal{G}_2}}$ be the $\mathcal{F}c\vec{\mathcal{G}}$-topologies formed by $\vec{\mathcal{G}}_1$ and $\vec{\mathcal{G}}_2$, respectively. Therefore, (N', $\mathfrak{I}_{\mathcal{F}c\vec{\mathcal{G}_1}}$) and (N", $\mathfrak{I}_{\mathcal{F}c\vec{\mathcal{G}_2}}$) are the $\mathcal{F}c\vec{\mathcal{G}}$ -topological spaces. To prove f : N' →N" is a homeomorphism.

(i) Clearly f is bijective.

(ii) Claim: f is continuous.

Let $\mathcal{S}_{\mathcal{F}c}$ be the subbasis of $\mathfrak{I}_{\mathcal{F}c\vec{\mathcal{G}_2}}$. By definition, $\mathcal{S}_{\mathcal{F}c}$ = {$\mathcal{F}c_{v''}$: v" ϵ N"($\vec{\mathcal{G}}_2$)}, where $\mathcal{F}c_{v''}$ = <A> = {w": w" ϵN" and (v", w") ϵL"}. By the definition of graph isomorphism, f: N' →N" is a bijection such that (v', w') ϵL' ⇔ (f(v'), f(w')) ϵ L". Let f(v') = v" and f(w') = w". Now, w" ϵ $\mathcal{F}c_{v''}$ ⇔ (v", w") ϵ L" ⇔ (v', w') ϵ L' ⇔ w' ϵ$\mathcal{F}c_{v'}$. Therefore, $\mathcal{F}c_{v'}$ = <A> = {w': w' ϵN' and (v', w') ϵL'}. Hence, $f^{-1}(\mathcal{F}c_{v''}) = \mathcal{F}c_{v'}$. Since $\mathcal{F}c_{v'}$ is a member of subbasis of $\mathfrak{I}_{\mathcal{F}c\vec{\mathcal{G}_1}}$, it is a $\mathfrak{I}_{\mathcal{F}c\vec{\mathcal{G}_1}}$ open subset of N'. Hence, the inverse image of a member of subbasis of $\mathfrak{I}_{\mathcal{F}c\vec{\mathcal{G}_2}}$ is an $\mathfrak{I}_{\mathcal{F}c\vec{\mathcal{G}_1}}$ open subset of N'. Therefore, f is continuous.

(iii)Claim: f is open.

Let $\mathcal{F}c_{v'}$ = {w': w' ϵN' and (v', w') ϵ L'} be a member of the subasis of $\mathfrak{I}_{\mathcal{F}c\vec{\mathcal{G}_1}}$. Clearly, it is an $\mathfrak{I}_{\mathcal{F}c\vec{\mathcal{G}_1}}$-open set. Let w' ϵ$\mathcal{F}c_{v'}$ ⇔ (v', w') ϵ L' ⇔ (v", w") ϵ L" ⇔ w" ϵ $\mathcal{F}c_{v''}$. Therefore, $\mathcal{F}c_{v''}$ = {w": w" ϵN" and (v", w") ϵ L"}. Hence $f(\mathcal{F}c_{v'}) = \mathcal{F}c_{v''}$. Clearly, $\mathcal{F}c_{v''}$ is a member of the subbasis of $\mathfrak{I}_{\mathcal{F}\vec{\mathcal{G}_2}}$. Also, it is an $\mathfrak{I}_{\mathcal{F}\vec{\mathcal{G}_2}}$-open set. So, image of every $\mathfrak{I}_{\mathcal{F}c\vec{\mathcal{G}_1}}$-open set is $\mathfrak{I}_{\mathcal{F}c\vec{\mathcal{G}_2}}$ -open. Therefore, f is open. Hence,$\mathfrak{I}_{\mathcal{F}c\vec{\mathcal{G}_1}}$ and $\mathfrak{I}_{\mathcal{F}c\vec{\mathcal{G}_2}}$ are homeomorphic.

4.Topology formed from Intuitionistic Crisp $\vec{\mathcal{G}}$-Set ($\mathfrak{T}c\vec{\mathcal{G}}$ - set)

Definition 4.1 Let $\vec{\mathcal{G}}$ be a D-graph, whose nodes are not isolated, the intuitionistic crisp$\vec{\mathcal{G}}$-set $\mathfrak{T}c_v$= <N, ζ_1, ζ_2>, where ζ_1-the set of in-neighbours and ζ_2-the set of out-neighbours of v respectively with the condition $\zeta_1 \cap \zeta_2$ = Φ.

Definition 4.2 In a D-graph $\vec{\mathcal{G}}$, whose nodes are not isolated, the set $\mathcal{S}_{\mathfrak{T}c}$ = {G : G ϵ $\mathfrak{T}c_v$, ∀ v ϵN($\vec{\mathcal{G}}$)} forms a sub basis for a $\mathfrak{T}c\vec{\mathcal{G}}$ -topology$\mathfrak{I}_{\mathfrak{T}c\vec{\mathcal{G}}}$. Also, (N, $\mathfrak{I}_{\mathfrak{T}c\vec{\mathcal{G}}}$) is called $\mathfrak{T}c\vec{\mathcal{G}}$-topological space on N.

Example 4.3 Consider a D-graph $\vec{\mathcal{G}}$ = (N, L) with N = {ς, ε, ϱ, φ}.

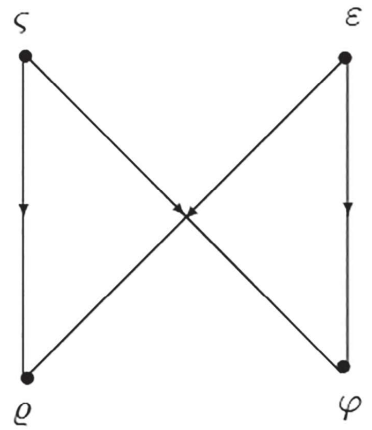

Figure 2. Graph for $\mathfrak{Tc}\vec{\mathcal{G}}$-topological space

Now, the $\mathfrak{Tc}_v$ sets for the vertices ς, ε, ϱ, φ are as follows:

$\mathfrak{Tc}_ς$= <N, Φ, {ϱ, φ}>; $\mathfrak{Tc}ε$= <N, Φ, {ϱ, φ}>;

$\mathfrak{Tc}_ϱ$= <N, {ς, ε}, Φ>; $\mathfrak{Tc}φ$= <N, {ς, ε}, Φ>.

So, $S_{\mathfrak{Tc}}$ = {Φ, {ς, ε}, {ϱ, φ}, N} forms a sub basis.

Hence, $\mathfrak{I}_{\mathfrak{Tc}\vec{\mathcal{G}}}$ = {Φ, {ς, ε}, {ϱ, φ}, N} is $\mathfrak{T}c\vec{\mathcal{G}}$-topological space on N.

Theorem 4.4 Every directed path graph generates a discrete $\mathfrak{Tc}\vec{\mathcal{G}}$ -topology.

Proof. Let $(\overrightarrow{P_m})$= (N, L) be a directed path graph with N={n_1, n_2, …, n_m}, L = {l_1, l_2, …, l_{m-1}}. Clearly each link l_i contribute an in-neighbour of n_{i+1} and an out-neighbour of n_i ∀ i = 1, 2, …, (m-1). Hence $\mathcal{S}_{\mathfrak{Tc}}$ = {{n_1}, {n_2}, …, {n_m}}. Clearly this subbasis generates a discrete $\mathfrak{Tc}\vec{\mathcal{G}}$ -topology on N.

Theorem 4.5 Let $K_{X,Y}$ = (N, L) be a complete bipartite D-graph with X = {u_1, u_2, …, u_n}, Y = {v_1, v_2, …, v_m} and in degree of u_i = 0, ∀ u_i and out degree of v_i = 0, ∀ v_i then the $\mathfrak{Tc}\vec{\mathcal{G}}$ -topology on N is quasi-discrete.

Proof. Since $K_{X,Y}$ is a complete bipartite D-graph, X U Y = N and X ∩ Y = Φ. Also, for every u_i ϵ X, $\mathfrak{T}c_{u_i}$ = {Φ, Y}, ∀ u_i and $\mathfrak{T}c_{v_j}$

= {X, Φ}, ∀ v_j. Therefore, $\mathcal{S}_{\mathfrak{T}c}$ = {φ, X, Y}. The topology generated by this subbasis is $\Im_{\mathfrak{T}c\vec{\mathcal{G}}}$ = {φ, X, Y, N}. Hence, the $\mathfrak{T}c\vec{\mathcal{G}}$-topology on N is quasi-discrete.

Remark 4.6 Any D-graph without isolated vertices and its converse graph generates same $\mathfrak{T}c\vec{\mathcal{G}}$-topology.

5. Topology generated by Neutrosophic Crisp $\vec{\mathcal{G}}$ Set ($\mathcal{N}c\vec{\mathcal{G}}$ - set)

Definition 5.1 Let $\vec{\mathcal{G}}$ = (N, L) be a D-graph whose nodes are not isolated. Then, the neutrosophic crisp$\vec{\mathcal{G}}$ - set $\mathcal{N}c_v$ = <ξ_1, ξ_2, ξ_3>, where ξ_1, ξ_3 are the set of in-neighbours, out-neighbours of v, respectively and ξ_2 = {u ϵ N : v is reachable from u also v and u are non-adjacent nodes} with the condition that $\xi_1 \cap \xi_2 = \Phi$, $\xi_1 \cap \xi_3 = \Phi$ and $\xi_2 \cap \xi_3 = \Phi$.

Definition 5.2 Let $\vec{\mathcal{G}}$ = (N, L) be a D-graph whose nodes are not isolated. Then, $\mathcal{S}_{\mathcal{N}c}$ = {G : G ϵ $\mathcal{N}c_v$: ∀v ϵN($\vec{\mathcal{G}}$)} forms a subbasis for a $\mathcal{N}c\vec{\mathcal{G}}$-topology $\Im_{\mathcal{N}c\vec{\mathcal{G}}}$. Also, (N, $\Im_{\mathcal{N}c\vec{\mathcal{G}}}$) is called $\mathcal{N}c\vec{\mathcal{G}}$-topological space on N.

Example 5.3 Consider a D-graph $\vec{\mathcal{G}}$ = (N, L) with N = {ς, ε, ϱ, φ}.

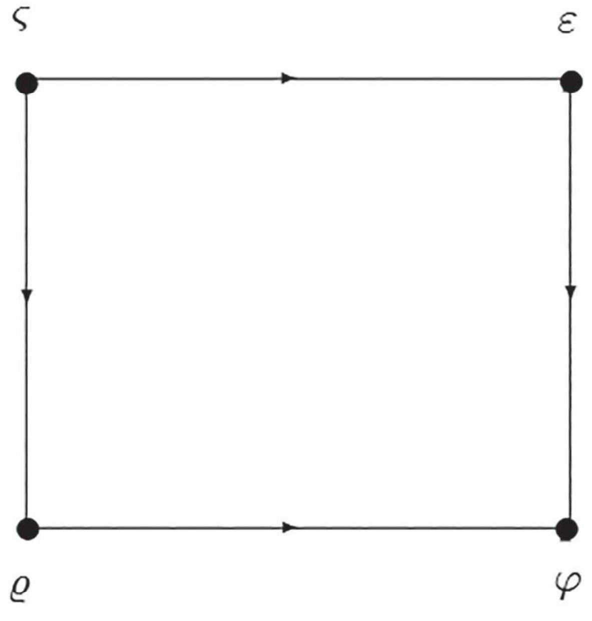

Figure 3. Graph for $\mathcal{N}c\vec{\mathcal{G}}$-topological space.

Now, the $\mathcal{N}c_v$ sets for the vertices ς, ε, ϱ, φ are as follows:

$Nc_ς$ = <φ, φ, {ε, ϱ}> ; $Nc_ε$ = <{ς}, φ, {φ}>

$Nc_ϱ$ = <{ς}, φ, {φ}> ; $Nc_φ$ = <{ ε, ϱ}, {ς}, φ>

So, $\mathcal{S}_{\mathcal{N}c}$ = {φ, {ς}, {φ}, {ε, ϱ}} forms a subbasis.

Hence, $\Im_{\mathcal{N}c\vec{\mathcal{G}}}$ = {φ, {ς}, {φ}, {ε, ϱ}, {ς, φ}, {ς, ε, ϱ}, {ε, ϱ, φ}, N} is a $\mathcal{N}c\vec{\mathcal{G}}$-- space on N.

Proposition 5.4 Every directed Euler graph generates a discrete $\mathcal{N}c\vec{\mathcal{G}}$ -topology.

Proof. Let $\vec{\mathcal{G}}$ a be a directed Euler graph. There is a directed path between every pair of nodes in $\vec{\mathcal{G}}$. Now, the basis obtained by $\mathcal{S}_{\mathcal{N}c}$ contains all singletons. Hence, it will generate discrete $\mathcal{N}c\vec{\mathcal{G}}$-topology.

Proposition 5.5 Every directed Hamiltonian graph generates discrete $\mathcal{N}c\vec{\mathcal{G}}$-topology.

Proof. Let $\vec{\mathcal{G}}$ be a directed Hamiltonian graph. There is a spanning cycle in $\vec{\mathcal{G}}$. Now, the basis obtained by $\mathcal{S}_{\mathcal{N}c}$ contains all singletons. Hence, it will generate discrete $\mathcal{N}c\vec{\mathcal{G}}$-topology.

Theorem 5.6 Any connected D-graph generates $\Im_{\mathfrak{T}c\vec{\mathcal{G}}}$ and $\Im_{\mathcal{N}c\vec{\mathcal{G}}}$. Also, $\Im_{\mathfrak{T}c\vec{\mathcal{G}}} \subseteq \Im_{\mathcal{N}c\vec{\mathcal{G}}}$.

Proof. Let $\vec{\mathcal{G}}$ = (N, L) be a connected D-graph. Here, $\vec{\mathcal{G}}$ has no isolated node, {G : G ϵ $\mathfrak{T}c_v$: ∀ v ϵN($\vec{\mathcal{G}}$)} and {H : H ϵ $\mathcal{N}c_v$: ∀v ϵN($\vec{\mathcal{G}}$)} forms a subbasis $\mathcal{S}_{\mathfrak{T}c}$ and $\mathcal{S}_{\mathcal{N}c}$ respectively. Let G ϵ$\mathcal{S}_{\mathfrak{T}c}$. Then, by definition G ϵ$\mathcal{S}_{\mathcal{N}c}$. Therefore,$\mathcal{S}_{\mathfrak{T}c} \subseteq \mathcal{S}_{\mathcal{N}c}$. Now, $\mathcal{S}_{\mathfrak{T}c}$ generates $\Im_{\mathfrak{T}c\vec{\mathcal{G}}}$ and $\mathcal{S}_{\mathcal{N}c}$ generates $\Im_{\mathcal{N}c\vec{\mathcal{G}}}$. This implies that $\Im_{\mathfrak{T}c\vec{\mathcal{G}}} \subseteq \Im_{\mathcal{N}c\vec{\mathcal{G}}}$.

Remark 5.7 All connected D-graphs need not generate $\Im_{\mathcal{F}c\overrightarrow{\mathcal{G}_1}}$. Suppose there exists some vertex v_i such that O(v_i) = Φ or I(v_i) = Φ. (i.e) $v_i \notin \mathcal{F}c_v$, ∀ v ϵ N. Hence, {$\mathcal{F}$cv : v ϵ N($\vec{\mathcal{G}}$)} does not form a subbasis $\mathcal{S}\mathcal{F}$c.

Theorem 5.8 For a strongly connected D-graph $\Im_{\mathcal{F}c\vec{\mathcal{G}}} \subseteq \Im_{\mathfrak{T}c\vec{\mathcal{G}}} \subseteq \Im_{\mathcal{N}c\vec{\mathcal{G}}}$.

Proof. Let $\vec{\mathcal{G}}$ = (N, L) be a strongly connected D-graph. Then, O(v_i) and I(v_i) ≠ Φ ∀ v_i ϵ N($\vec{\mathcal{G}}$). Therefore, each vertex v_i ϵ $\mathcal{F}c_{v_j}$ for some i ≠ j. Now {$\mathcal{F}_v$: v ϵ N($\vec{\mathcal{G}}$)} forms a subbasis $\mathcal{S}_{\mathcal{F}c}$. (i.e.) U$\mathcal{F}c_v$ = N($\vec{\mathcal{G}}$). Hence, it generates $\Im_{\mathcal{F}c\vec{\mathcal{G}}}$. Let $\mathcal{S}_{\mathcal{F}c}$ and $\mathcal{S}_{\mathfrak{T}c}$ be the subbases of $\Im_{\mathcal{F}c\vec{\mathcal{G}}}$ and $\Im_{\mathfrak{T}c\vec{\mathcal{G}}}$ respectively. Let $\mathcal{F}c_{v_i} \epsilon \mathcal{S}_{\mathcal{F}c}$. Then, by definition, $\mathcal{F}c_{v_i} \epsilon \mathcal{S}_{\mathfrak{T}c}$. Therefore $\Im_{\mathcal{F}c\vec{\mathcal{G}}} \subseteq \Im_{\mathfrak{T}c\vec{\mathcal{G}}}$. Since $\vec{\mathcal{G}}$ is connected, by theorem 5.6, $\Im_{\mathfrak{T}c\vec{\mathcal{G}}} \subseteq \Im_{\mathcal{N}c\vec{\mathcal{G}}}$. Hence, $\Im_{\mathcal{F}c\vec{\mathcal{G}}} \subseteq \Im_{\mathfrak{T}c\vec{\mathcal{G}}} \subseteq \Im_{\mathcal{N}c\vec{\mathcal{G}}}$.

Proposition 5.9 For any tournament $\Im_{\mathfrak{T}c\vec{\mathcal{G}}} = \Im_{\mathcal{N}c\vec{\mathcal{G}}}$.

Proof. In a tournament every node is either an in-neighbour or an out-neighbour of all other nodes. By definition of $\mathcal{S}_{\mathfrak{T}c}$ and $\mathcal{S}_{\mathcal{N}c}$, clearly $\mathcal{S}_{\mathfrak{T}c} = \mathcal{S}_{\mathcal{N}c}$. Hence, it generates a unique topology. Hence, $\Im_{\mathfrak{T}c\vec{\mathcal{G}}} = \Im_{\mathcal{N}c\vec{\mathcal{G}}}$.

6.Separation Axioms on $\mathcal{F}c\vec{\mathcal{G}}$ / $\mathfrak{T}c\vec{\mathcal{G}}$ / $\mathcal{N}c\vec{\mathcal{G}}$ Spaces

Definition 6.1 Let $\vec{\mathcal{G}}$ = (N, L) be a D-graph and (N, $\Im_{\mathcal{F}c\vec{\mathcal{G}}}(\Im_{\mathfrak{T}c\vec{\mathcal{G}}} / \Im_{\mathcal{N}c\vec{\mathcal{G}}})$) be a $\mathcal{F}c\vec{\mathcal{G}}$ ($\mathfrak{T}c\vec{\mathcal{G}}$ /$\mathcal{N}c\vec{\mathcal{G}}$)-topological space. Then, (N, $\Im_{\mathcal{F}c\vec{\mathcal{G}}}(\Im_{\mathfrak{T}c\vec{\mathcal{G}}} / \Im_{\mathcal{N}c\vec{\mathcal{G}}})$) is a $\mathcal{F}c\vec{\mathcal{G}}$ ($\mathfrak{T}c\vec{\mathcal{G}}$ /$\mathcal{N}c\vec{\mathcal{G}}$)- T_0 space if for every u, v ϵ N($\vec{\mathcal{G}}$), u $\neq$v, there exist a $\mathcal{F}c\vec{\mathcal{G}}$ ($\mathfrak{T}c\vec{\mathcal{G}}$ /$\mathcal{N}c\vec{\mathcal{G}}$)-open set contains either v or u.

Example 6.2 Consider a D-graph $\vec{\mathcal{G}}$ = (N, L) with N = {ς, ε, ϱ, φ}.

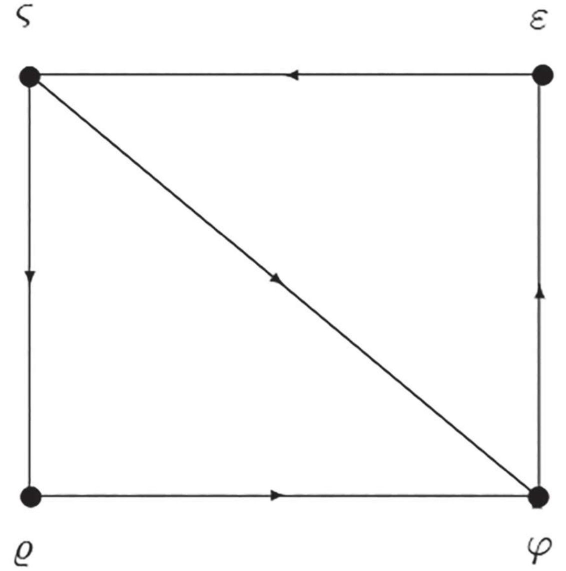

Figure 4. Graph for $\mathcal{F}c\vec{\mathcal{G}}$ -T_0 space

Now, $\mathcal{S}_{\mathcal{F}c}$ = {{ε}, {φ}, {ς}, {ς, ϱ}} forms a subbasis.

Hence, , $\Im_{\mathcal{F}\vec{\mathcal{G}}}$ = {Φ, {ς}, {ε}, {φ}, {ς, ε}, {ς, ϱ}, {ς, φ}, {ε, φ}, {ς, ε, ϱ}, {ς, ϱ, φ}, {ς, ε, φ}, N}. Therefore, (N, $\Im_{\mathcal{F}c\vec{\mathcal{G}}}$) is $\mathcal{F}c\vec{\mathcal{G}}$ -T_0 space.

Example 6.3 Consider a D-graph $\vec{\mathcal{G}}$ = (N, L) with N = {ς, ε, ϱ, φ}.

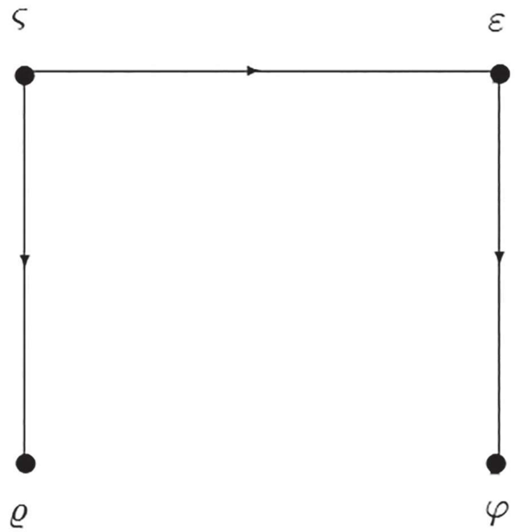

Figure 5. Graph for $\mathfrak{T}c\vec{\mathcal{G}}$-$T_0$ space

Now, $\mathcal{S}_{\mathfrak{T}c}$ = {Φ, {ς}, {ε}, {φ}, {ε, ϱ}, N} forms a sub basis.

Hence, $\Im_{\mathfrak{T}c\vec{\mathcal{G}}}$ = {Φ, {ς}, {ε}, {φ}, {ς, ε}, {ς, φ}, {ε, ϱ}, {ε, φ}, {ς, ε, ϱ}, {ς, ε, φ}, {ε, ϱ, φ}, N}. Therefore, (N, $\Im_{\mathfrak{T}c\vec{\mathcal{G}}}$) is $\mathfrak{T}c\vec{\mathcal{G}}$-$T_0$ space.

Example 6.4 Consider a D-graph $\vec{\mathcal{G}}$ = (N, L) with N = {ς, ε, ϱ, φ}.

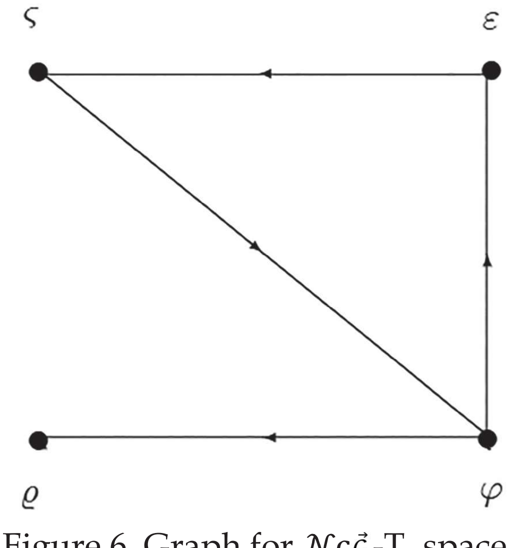

Figure 6. Graph for $\mathcal{N}c\vec{\mathcal{G}}$-$T_0$ space.

Now, $\mathcal{S}_{\mathcal{N}c}$ = {Φ, {ς}, {ε}, {φ}, {ς, ε}, {ε, ϱ}} forms a subbasis.

Hence, $\mathfrak{I}_{\mathcal{N}\vec{\mathcal{G}}}$ = {Φ, {ς}, {ε}, {φ}, {ς, ε}, {ς, φ}, {ε, ϱ}, {ε, φ}, {ς, ε, ϱ}, {ς, ε, φ}, {ε, ϱ, φ}, N}. Therefore, (N, $\mathfrak{I}_{\mathcal{N}c\vec{\mathcal{G}}}$) is $\mathcal{N}c\vec{\mathcal{G}}$ -T_0 space.

Definition 6.5 Let (N, $\mathfrak{I}_{\mathcal{F}c\vec{\mathcal{G}_1}}$ ($\mathfrak{I}_{\mathfrak{T}c\vec{\mathcal{G}}}$ / $\mathfrak{I}_{\mathcal{N}c\vec{\mathcal{G}}}$)) be a $\mathcal{F}c\vec{\mathcal{G}}$($\mathfrak{T}c\vec{\mathcal{G}}$ / $\mathcal{N}c\vec{\mathcal{G}}$) -topological space on the node set of a D-graph $\vec{\mathcal{G}}$ = (N, L). Then, (N, $\mathfrak{I}_{\mathcal{F}c\vec{\mathcal{G}}}$ ($\mathfrak{I}_{\mathfrak{T}c\vec{\mathcal{G}}}$ / $\mathfrak{I}_{\mathcal{N}c\vec{\mathcal{G}}}$)) is a $\mathcal{F}c\vec{\mathcal{G}}$($\mathfrak{T}c\vec{\mathcal{G}}$ / $\mathcal{N}c\vec{\mathcal{G}}$)- T_1 space if for every distinct nodes a, b ϵ N($\vec{\mathcal{G}}$), there exists two distinct $\mathcal{F}c\vec{\mathcal{G}}$($\mathfrak{T}c\vec{\mathcal{G}}$ / $\mathcal{N}c\vec{\mathcal{G}}$)-open sets containing one but not other.

Definition 6.6 Let (N, $\mathfrak{I}_{\mathcal{F}c\vec{\mathcal{G}}}$ ($\mathfrak{I}_{\mathfrak{T}c\vec{\mathcal{G}}}$ / $\mathfrak{I}_{\mathcal{N}c\vec{\mathcal{G}}}$)) be a $\mathcal{F}c\vec{\mathcal{G}}$($\mathfrak{T}c\vec{\mathcal{G}}$ / $\mathcal{N}c\vec{\mathcal{G}}$)-topological space on the node set of a D-graph $\vec{\mathcal{G}}$ = (N, L). Then, (N, $\mathfrak{I}_{\mathcal{F}c\vec{\mathcal{G}}}$ ($\mathfrak{I}_{\mathfrak{T}c\vec{\mathcal{G}}}$ / $\mathfrak{I}_{\mathcal{N}c\vec{\mathcal{G}}}$)) is a $\mathcal{F}c\vec{\mathcal{G}}$($\mathfrak{T}c\vec{\mathcal{G}}$ / $\mathcal{N}c\vec{\mathcal{G}}$)- T_2 space if for every distinct nodes a, b ϵ N($\vec{\mathcal{G}}$), there are two disjoint $\mathcal{F}c\vec{\mathcal{G}}$($\mathfrak{T}c\vec{\mathcal{G}}$ / $\mathcal{N}c\vec{\mathcal{G}}$)-open sets ℙ and ℚ such that a ϵ ℙ, b ϵ ℚ.

Proposition 6.7 Every directed cycle $\overrightarrow{C_n}$, n ≥ 3, generates $\mathcal{F}c\vec{\mathcal{G}}$-$T_1$ space and $\mathcal{F}c\vec{\mathcal{G}}$-$T_2$ space.

Proof. By Theorem 3.7, every directed cycle $\overrightarrow{C_n}$ generates discrete $\mathcal{F}c\vec{\mathcal{G}}$-topology. Hence, it is $\mathcal{F}c\vec{\mathcal{G}}$-$T_1$ space and $\mathcal{F}c\vec{\mathcal{G}}$-$T_2$ space.

Proposition 6.8 Every directed path graph $\overrightarrow{P_n}$ generates $\mathfrak{T}c\vec{\mathcal{G}}$-$T_1$ space and $\mathfrak{T}c\vec{\mathcal{G}}$-$T_2$ space.

Proof. By Theorem 4.4, every directed path graph $\overrightarrow{P_n}$ generates discrete $\mathfrak{T}c\vec{\mathcal{G}}$-topology. Hence, it is $\mathfrak{T}c\vec{\mathcal{G}}$-$T_1$ space and $\mathfrak{T}c\vec{\mathcal{G}}$-$T_2$ space.

Remark 6.9 Every directed cycle $\overrightarrow{C_n}$, n ≥ 3 and every directed path graph $\overrightarrow{P_n}$ generates $\mathcal{N}c\vec{\mathcal{G}}$-$T_1$ space and $\mathcal{N}c\vec{\mathcal{G}}$-$T_2$ space.

7. T_i-Graphs (i = 0, 1, 2)

Definition 7.1 A simple connected D-graph $\vec{\mathcal{G}}$ is said to be T_0-Graph if for any two different nodes u and v of $\vec{\mathcal{G}}$ either there exist distinct in-neighbours or distinct out-neighbours.

Definition 7.2 A simple connected D-graph $\vec{\mathcal{G}}$ is said to be T_1-Graph if for any two different nodes u and v of $\vec{\mathcal{G}}$ there exist both distinct in-neighbours and distinct out-neighbours.

Definition 7.3 A simple connected D-graph $\vec{\mathcal{G}}$ is said to be T_2-Graph if for any two different nodes u and v of $\vec{\mathcal{G}}$ there exist both disjoint in-neighbourhood and disjoint out-neighbourhood.

Example 7.4 Consider the D-graph in Example 6.2 in which the set of in-neighbours of each node are distinct. Hence, it is a T_0-Graph.

Example 7.5 Consider the D-graphs in Example 6.3 and Example 6.4, in both of which the set of out-neighbours of each node are distinct. Hence, they are T_0-Graphs.

Remark 7.6 Every T_0-Graph which induces $\mathcal{F}c\vec{\mathcal{G}}$-$T_0$ space also induces $\mathfrak{T}c\vec{\mathcal{G}}$-$T_0$space and $\mathcal{N}c\vec{\mathcal{G}}$-$T_0$ space.

Remark 7.7 The following example shows that, a D-graph $\vec{\mathcal{G}}$ which is not a T_0-Graph then the corresponding $\mathfrak{T}c\vec{\mathcal{G}}/\mathcal{N}\ c\vec{\mathcal{G}}$-topologies induced by $\vec{\mathcal{G}}$ is also not a T_0-space.

Example 7.8 Consider the D-graph in Example 5.3, which is not a T_0-graph. The $\mathfrak{T}c\vec{\mathcal{G}}$-topology and $\mathcal{N}c\vec{\mathcal{G}}$-topology induced by it are also not $\mathfrak{T}c\vec{\mathcal{G}}$-$T_0$ space and $\mathcal{N}c\vec{\mathcal{G}}$-$T_0$ space respectively.

Example 7.9 Consider the D-graph in Example 6.2, in which O(v) and I(v) of each node v are distinct. Hence, it is a T_1-Graph.

Proposition 7.10 Every T_1-Graph is a T_0-Graph.

Proof: Obvious, from the definition of T_0-Graph and T_1-Graph.

Remark 7.11 The concepts T_1-Graph and $\mathcal{F}c\vec{\mathcal{G}}$ / $\mathfrak{T}c\vec{\mathcal{G}}/\mathcal{N}c\vec{\mathcal{G}}$- T_1 Spaces are independent.

Example 7.12 Consider the D-graph in Example 6.2, which is T_1-Graph but the $\mathcal{F}c\vec{\mathcal{G}}$-topology induced by it is not $\mathcal{F}c\vec{\mathcal{G}}$-$T_1$ space.

Example 7.13 Consider the D-graph $\vec{\mathcal{G}}$ = (N, L) with N = {ς, ε, ϱ, φ}.

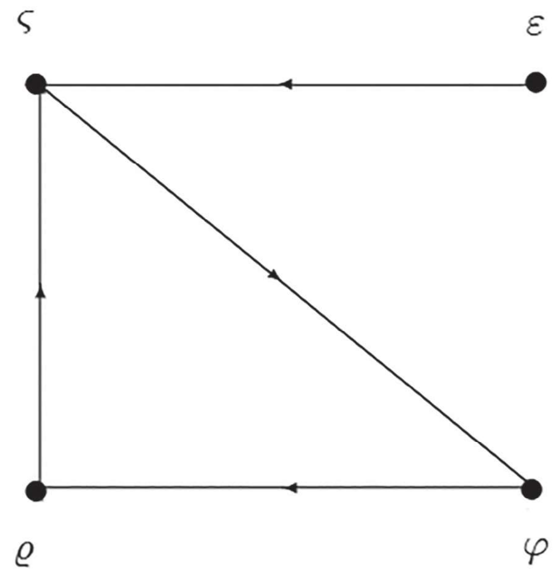

Figure 7. Not a T_1-graph

This is not a T_1-Graph, since the out-neighbours of ε and ϱ are same; but the $\mathcal{N}c\vec{\mathcal{G}}$-topology induced by it is a $\mathcal{N}c\vec{\mathcal{G}}$-$T_1$ space.

Example 7.14 Directed cycle $\overrightarrow{C_3}$ is a T_2-Graph.

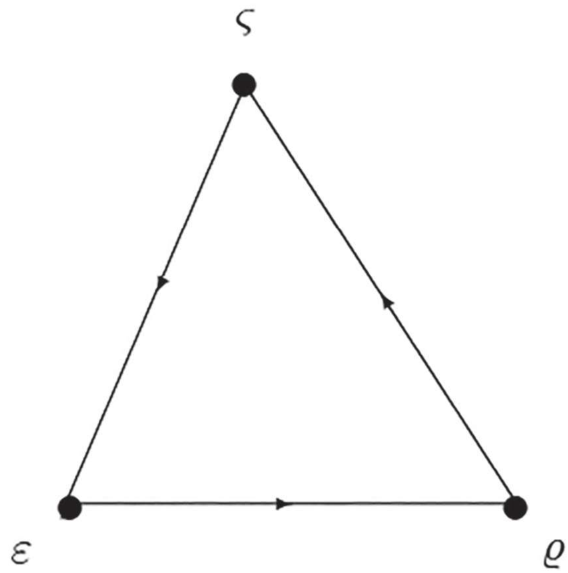

Figure 8. Cycle $\overrightarrow{C_3}$ – T_2-graph

O(ς) = {ε}, O(ε) = {ϱ} and O(ϱ) = {ς}. I(ς) = {ϱ}, I(ε) = {ς} and I(ϱ) = {ε}.

Since the in-neighbours and out-neighbours of every node are disjoint, $\overrightarrow{C_3}$ is a T_2-Graph.

Proposition 7.15 Directed cycle $\overrightarrow{C_n}$ is a T_2-Graph for n ≥ 3.

Proof. Consider a cycle graph with 'n' nodes.

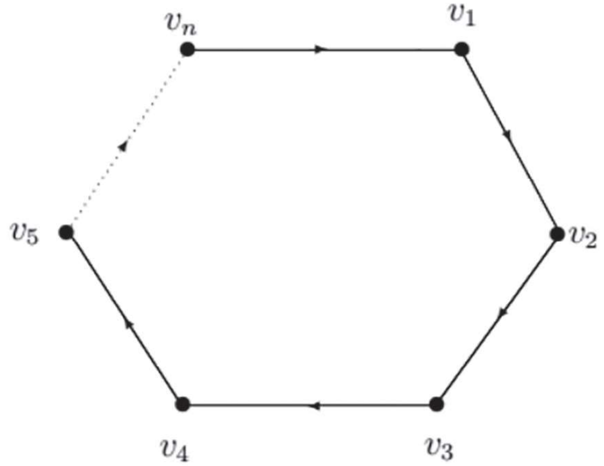

Figure 9. Cycle $\overrightarrow{C_n}$ – T_2-graph

Now, $I(v_1) = \{v_n\}$ and $I(v_i) = \{v_{i-1}\}$ for i = 2, 3, …, n. Also, $O(v_n) = \{v_1\}$ and $O(v_i) = \{v_{i+1}\}$ for i = 1, 2, …, n-1. Hence, $I(v_i)$ and $O(v_i)$ are disjoint for all v_i, i = 1, 2, …, n. Therefore, directed cycle $\overrightarrow{C_n}$ is a T_2-Graph for n ≥ 3.

Remark 7.16 Directed cycle $\overrightarrow{C_n}$, n ≥ 3 is the only T_2-Graph which induces $\mathcal{F}c\vec{\mathcal{G}}$-$T_2$ space, $\mathfrak{T}c\vec{\mathcal{G}}$-$T_2$ space and $\mathcal{N}c\vec{\mathcal{G}}$-$T_2$ space.

Example 7.17 Directed path $\overrightarrow{P_n}$, is a T_2-Graph for n ≥ 2.

Remark 7.18 The T_2-Graph $\overrightarrow{P_n}$, , n ≥ 2 induces $\mathfrak{T}c\vec{\mathcal{G}}$-$T_2$ space and $\mathcal{N}c\vec{\mathcal{G}}$-$T_2$ space.

Proposition 7.19 Every T_2-Graph is a T_1-Graph.

Proof. Obvious from the definition of T_1-Graph and T_2-Graph.

Conclusion

In this paper, we have derived the topological spaces from directed graphs with the help of neutrosophic crisp set concept. On every vertex of a digraph, we have framed a new type of fuzzy crisp, intuitionistic crisp and neutrosophic crisp sets. Using these sets, we have generated topologies from simple digraphs and examined their characterisations. Separation axioms on digraphs are defined, relating the topological separation axioms generated by digraphs. This article may would lead to significant applications. In future, we may also extend this in some peculiar graphs.

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