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Thermal convection of a Oldroyd-B nanofluid with magnetic effect: Linear and weakly nonlinear analyses

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Abstract

Onset of convection in a horizontal layer with Oldroyd-B nanofluid investigated. The normal mode technique has been employed to work out the non dimensional governing equations and this leads to eigenvalue problem. The free-free boundary conditions have been considered. The analytical expressions of stationary and oscillatory Rayleigh numbers are obtained using one term Galerkin method. Critical values of Rayleigh number for the prescribed values of other parameters are obtained. From linear theory, it is proved that the Hartmann number and Prandtl number has stabilizing effect on the flow. Amplitude equation is derived in weakly nonlinear analysis. Heat transport is studied by calculating Nusselt number using amplitude. Keywords: Nanofluid, Thermal convection, Casson model, Linear analysis, nonlinear analysis.

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1. Introduction

Nanofluids are formed by dispersing a small quantity of metallic or non-metallic nanoparticles into a conventional base fluid. The term 'nanofluid' specifically denotes a liquid containing a suspension of solid particles at the

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nanometer scale. The term was coined by Choi[1]. The characteristic feature of nanofluids is thermal conductivity enhancement, a phenomenon observed by Masuda et al. [2].

This phenomenon suggests the possibility of using nanofluids in advanced nuclear systems by Buongiorno and Hu [3]. The Bnard problem (the onset of convection in a horizontal layer uniformly heated from below) for a nanofluid was studied by Tzou [4-5] and Nield and Kuznetsov[6] on the basis of the transport equations of Buongiorno [7]. The corresponding problem for flow in a porous medium (the HortonRogersLap wood problem) was studied by Nield and Kuznetsov [8] using the Darcy model.

Nowadays, the Casson fluid model is widely employed in the food industry, particularly by cocoa and chocolate manufacturers, to describe and analyze the rheological behavior of chocolate. Moreover, these days the Casson model is also used for developing the rheological model for human blood [9-12]. Some researchers [13-15] propounded that for blood flowing through small vessels, there is an erythrocyte-free plasma (Newtonian) layer adjacent to the vessel wall and a core layer of a suspension of all erythrocytes (non-Newtonian). It has been pointed out both by Scott Blair [16] and Iida [17] that though it is possible to model the blood flow by both Casson fluid model and HerschelBulkley fluid over the range where both models are valid, Casson fluid model is well suited and simple to apply for blood flow problems.

Many researchers [18-21] have used Casson fluid model for mathematical modelling of blood flow through narrow arteries at low shear rates for different flow situations. Therefore, it is reasonable to model the blood in the core region of the two-fluid blood flow system as a Casson fluid. In recent years, nanoparticles have found extensive applications in the treatment of various diseases. In particular, gold nanoparticles are utilized in cancer therapy due to their relatively larger size and strong energy absorption capacity. Moreover, nanoparticles influence the heat transfer mechanism between the heart and the body surface through blood convection. Consequently, the study of convective instability of blood in the presence of nanoparticles plays a vital role in the medical field, contributing to advancements in healthcare practices.

As observed in the literature magnetic effect on Casson nanofluid is not studied. Moreover, the above studies concern with linear analysis of Casson nanofluid. The present article have been considered conservation equations for blood flow which are modeled by Casson nanofluid. The free-free boundary conditions have been considered. The equations have been solved by one term Galerkin method. The Rayleigh number is expressed in terms of various parameters.

2. Mathematical formulation

Consider a heated, infinitely thin, horizontal layer of Oldroyd-B nanofluid with thickness d that is confined by the planes z = 0 and z = d. The volumetric fraction ϕ and temperature T of nanoparticles are assumed to be T_0 and ϕ_0 at z = 0 and T_1 and ϕ_1 at z = d, respectively ($T_0 > T_1$) (see Fig. 1). The assumed reference temperature is T_1.

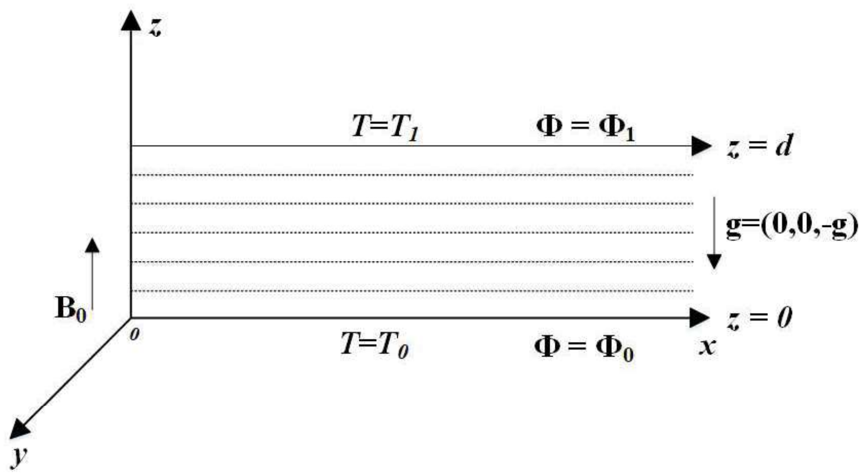

Figure 1: Physical Configuration.

We consider stress tensor of Casson fluid as

$$\tau_{ij} = \begin{cases} 2\left(\mu_B + \frac{p_z}{\sqrt{2\pi}}\right) e_{ij} \text{ if } \pi < \pi_c; \\ 2\left(\mu_B + \frac{p_z}{\sqrt{2\pi_C}}\right) e_{ij} \text{ if } \pi > \pi_c \end{cases} \tag{1}$$

Where

$$\begin{cases} \mu_B - dynamic\ viscosity, p_z - yield\ shear\ stress, \\ e_{ij} - rate\ of\ deformation\ tensor, \qquad \pi = e_{ij}e_{ij}, \\ \pi_c - critical\ value\ of\ \pi \end{cases}$$

The governing equations are [22-24]:

$$\nabla . \mathrm{V} = 0 \tag{2}$$

$$\left(1 + \bar{\lambda}_1 \frac{\partial}{\partial t}\right)\left(\rho_0 \frac{\partial V}{\partial t} + \rho_0 (V.\nabla)V - [\phi\rho_p + (1-\phi)\rho_0(1 - \beta(T - T_1))]g + \nabla P - \sigma_1 (V \times B_o \hat{e}_z) = \mu\left(1 + \bar{\lambda}_2 \frac{\partial}{\partial t}\right)\nabla^2 V, \tag{3}$$

$$\frac{\partial}{\partial t} + (V . \nabla))\phi = D_B \nabla^2 \phi + \frac{D_T}{T_1}\nabla^2 T, \tag{4}$$

$$(\rho c)_f \left[\frac{\partial T}{\partial t} + V . \nabla T\right] = k\nabla^2 T + (\rho c)_p \left[D_b \nabla\phi . \nabla T + \frac{D_t}{T_0}\nabla T . \nabla T\right], \tag{5}$$

where

$$\begin{cases} \mathrm{V} = (u^*, \quad v^*, \quad w^*) = Velocity, t = Time, P = Pressure, \\ \phi = Volume\ fraction, \quad T = Temperature \\ \mu = Viscosity, \\ g = Gravity, \\ k = Thermal\ conductivity, \rho = Density, \\ \rho c = Heat\ capacity, Q_o = Volumetric\ heat\ source. \\ D_b = Diffusion\ coefficient\ of\ Brownian, D_t = Thermophoresis\ diffusion\ coefficient \\ \bar{\lambda}_1 = \bar{\lambda}_2 = Relaxation\ times\ of\ the\ fluid \end{cases}$$

The boundary conditions are

$$T = T_0, \phi = \phi_0 \ at\ Z = 0, \tag{6}$$

$$T = T_1, \phi = \phi_1 \ at\ Z = d, \tag{7}$$

The basic state is described by

$$v_b = 0, \phi_b = 0, T_b = T_0 - \left(\frac{\Delta T}{d}\right) z \tag{8}$$

We define the following non-dimensional parameters:

$$(x', y', z') = \frac{1}{d}(x, y, z), \qquad t' = \frac{\alpha_f t}{d^2}, \qquad P' = \frac{d^2 P}{\mu \alpha_f},$$

$$(u', v', w') = \frac{d}{\alpha_f}(u, v, w), \qquad T' = \frac{T - T_1}{T_0 - T_1}, \qquad \phi' = \frac{\phi - \phi_0}{\phi_1 - \phi_0},$$

$$\alpha_f = \frac{k}{(\rho c_p)f},$$

where $\alpha = \frac{1}{\rho C}$.

Hence, Eqs. 2 – 5 are modified as

$$\nabla'.V' = 0, \tag{9}$$

$$\left(1 + \lambda_1 \frac{\partial}{\partial t}\right)\left(\frac{1}{P_r}\left(\frac{\partial V'}{\partial t'} + (V'.\nabla')V'\right) - \nabla' P' + R_m \hat{e}_z - RaT'\hat{e}_z + R_n \phi' \hat{e}_z - Ha^2[(V' \times \hat{e}_z) \times \hat{e}_z]\right) = \left(1 + \lambda_2 \frac{\partial}{\partial t}\right) \nabla'^2 V', \tag{10}$$

$$\left(\frac{\partial}{\partial t'} + (V'.\nabla')\right)\phi' = \frac{N_A}{Le}\nabla'^2 T' + \frac{1}{Le}\nabla'^2 \phi' \tag{11}$$

$$\left(\frac{\partial}{\partial t'} + (V'.\nabla')\right)T' = \nabla'^2 T' + \frac{N_B}{L_e}(\nabla' T'.\nabla'\phi') + \frac{N_A N_B}{Le}(\nabla' T'.\nabla' T'), \tag{12}$$

$$w = 0, D^2 W = 0, \phi = 0, T = 0: Z = 0, 1 \tag{13}$$

where

$$R_a = \rho_{f0} g K^3 \beta_1 \frac{(T_0^* - T_1^*)}{\mu \alpha_f}, Rayleigh\ number$$

$$L_e = \frac{\alpha_f}{D_b}, Lewis\ number$$

$$P_r = \frac{\mu}{\rho\alpha_f}, Prandtl\ number$$

$$N_b = (\rho c)_p \frac{(\phi_0^* - \phi_1^*)}{\rho c}, Modified\ particle\ density\ increment$$

$$N_a = \frac{D_t}{D_b}\frac{(T_0^* - T_1^*)}{T_1(\phi_1^* - \phi_0^*)}, Modified\ diffusivity\ ratio$$

$$R_m = \left(\frac{\rho_p\phi_0^* + \rho_{f0}(1-\phi_0^*)}{\mu\alpha_f}\right) gK^3, basic\ density\ Rayleigh\ number$$

$$R_n = \frac{(\rho_p - \rho_{f0})(\phi_1^* - \phi_0^*)}{\mu\alpha_f} gK^3, nanoparticle\ Rayleigh\ number$$

$$H\alpha^2 = \frac{\sigma_1 B_0^2 d^2}{\mu}, Hartmann\ number$$

$$\lambda_1 = \frac{\bar{\lambda}_1\alpha_f}{d^2}, stress\ relaxation\ number$$

$$\lambda_2 = \frac{\bar{\lambda}_2\alpha_f}{d^2}, strain\ retardation\ number$$

(14)

3. Linear stability analysis

To study the linear theory, we consider the linear parts of Eq. (13)-(16).

$$\left(1+\lambda_1\frac{\partial}{\partial t}\right)\left(\frac{1}{p_r}\frac{\partial v}{\partial t}+\nabla p+R_m e_z-R_a T e_z+R_n\phi e_z-Ha^2[(V'\times\hat{e}_z)\times\hat{e}_z]\right)=\left(1+\lambda_2\frac{\partial}{\partial t}\right)\nabla^2 v, \tag{15}$$

$$\frac{\partial T}{\partial t}=w+\nabla^2 T+\frac{N_b}{L_e}\nabla\phi.\nabla T+\frac{N_a N_b}{L_e}\nabla T.\nabla T\ , \tag{16}$$

$$\frac{\partial\phi}{\partial t}=\frac{1}{L_e}\nabla^2\phi+\frac{N_a}{L_e}\nabla^2 T, \tag{17}$$

$$w=0, D^2w=0, \phi=0, T=0\ at\ Z=0,$$

$$w=0,\ D^2w=0, \phi=0, T=0\ at\ Z=1. \tag{18}$$

Again taking the third component of double curl of 15

$$A_{11}\nabla^2 w-\left(1+\lambda_1\frac{\partial}{\partial t}\right)Ha^2D^2w+\left(1+\lambda_1\frac{\partial}{\partial t}\right)\nabla_h^2(RaT-Rn\phi)=0, \tag{19}$$

Where $A_{11}=\left(1+\lambda_2\frac{\partial}{\partial t}\right)\nabla^2-\left(1+\lambda_1\frac{\partial}{\partial t}\right)\frac{1}{Pr}\frac{\partial}{\partial t}$.

Let us consider the normal mode solution $(w,T,\phi)=(W,\theta,\phi)ze^{i(lx+my)+i\omega t}$ into the Eqs. 19, 16, and 17

$$\left(((1+\lambda_2 i\omega)(D^2-q^2)-(1+\lambda_1 i\omega)\frac{i\omega}{Pr}\right)(D^2-q^2)-(1+\lambda_1 i\omega)Ha^2D^2)W \tag{20}$$

$$+(1+\lambda_1 i\omega)(Rnq^2\phi-Raq^2\theta)=0, \tag{21}$$

$$\left(i\omega-(D^2-q^2)-\frac{Nb}{Le}D+2\frac{NaNb}{Le}D\right)\theta-W+D\phi=0, \tag{22}$$

$$\left(i\omega-\frac{1}{Le}(D^2-q^2)\right)\phi-\frac{Nb}{Le}(D^2-q^2)\theta=0. \tag{23}$$

$$W=\ D^W=\theta=\ \phi=0: z=0,1 \tag{24}$$

Boundary conditions chosen here allow one to apply the one-term Galerkin method to obtain the analytical expressions for steady and overstable Rayleigh numbers. We choose the solution in the form of

$$(W,\theta,\phi) = (W_0,\theta_0,\phi_0)\sin\Pi z. \qquad (25)$$

where W_0, θ_0, ϕ_0 are unknowns. On substituting the above solution in Eqs. 20 to 23, we get a system of equations in three unknowns coefficients W_o, θ_0, ϕ_0. After eliminating these unknown coefficients, we obtain the expression for Rayleigh number as

$$Ra = \frac{(a_1\omega^4 + a_2\omega^2 + a_3) + i\omega(b_1\omega^2 + b_2)}{Prq^2\delta^2(\lambda_1\omega^2 + 1)} \qquad (26)$$

Where,

$$\begin{cases} a_1 = -\delta^4\lambda_1^2, \\ a_2 = -NaPrq^2Rn\delta^2\lambda_1^2 + \delta^4(-1 + Ha^2\Pi^2Pr\lambda_1^2) + Pr\delta^6(\lambda_1 - \lambda_2) + Pr\delta^8\lambda_1\lambda_2, \\ a_3 = Pr\delta^2(-Naq^2Rn + Ha^2\Pi^2\delta^2 + \delta^6), \\ b_1 = \lambda_1\left(-LePrq^2Rn\lambda_1 + Ha^2\Pi^2Pr\delta^2\lambda_1 + \delta^6(\lambda_1 + Pr\lambda_2)\right), \\ b_2 = \left(-LePrq^2Rn + Ha^2\Pi^2Pr\delta^2\lambda_1 + \delta^6\left(1 + \mathrm{Pr}\left(1 + \delta^2(-\lambda_1 + \lambda_2)\right)\right)\right). \end{cases}$$

On substituting $\omega = 0$, one obtains the Ra for stationary convection as

$$Ra_{sc} = -RnNa + Ha^2\Pi^2\frac{\delta^2}{q^2} + \frac{\delta^6}{q^2} \qquad (27)$$

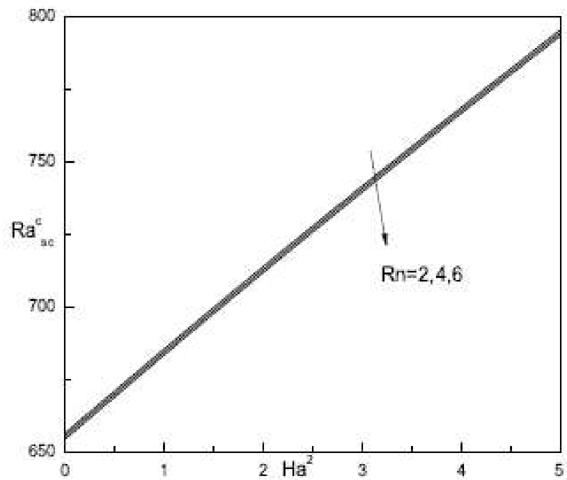

Figure 2: Change of $\boldsymbol{Ra_{sc}^c}$ with $H\boldsymbol{a}^{\mathbf{2}}$ for *Na* = 0.5.

For Newtonian liquids, without the Coriolis effect, the above formula becomes

$$Ra_{sc} = \frac{\delta^6}{q^2} \tag{28}$$

Which is well agreed with Chandrasekhar [25].

To determine the Rayleigh number for oscillatory convection, the roots of the imaginary part are first obtained. Substituting these roots into the real part of the Rayleigh number yields the critical *Ra* for oscillatory convection.

4. Weakly nonlinear analysis

To analyze the nature of convective motion, the weakly nonlinear theory is employed. For this purpose, we consider the non-dimensional equations including the nonlinear terms, given as:

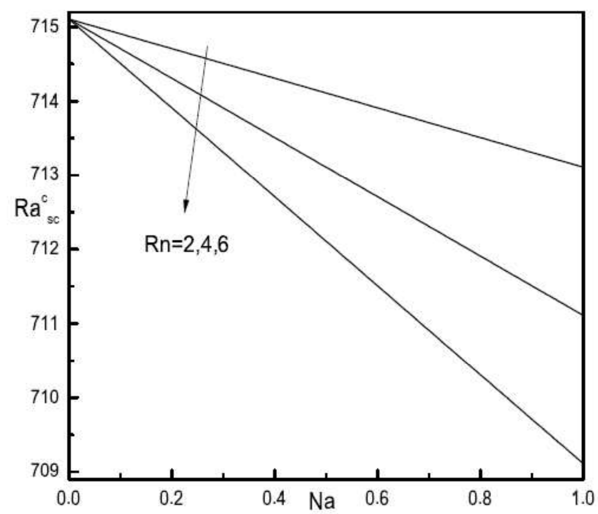

Figure 3: Change of $\boldsymbol{Ra^c_{sc}}$ with Na for $Ha^2 = 2$.

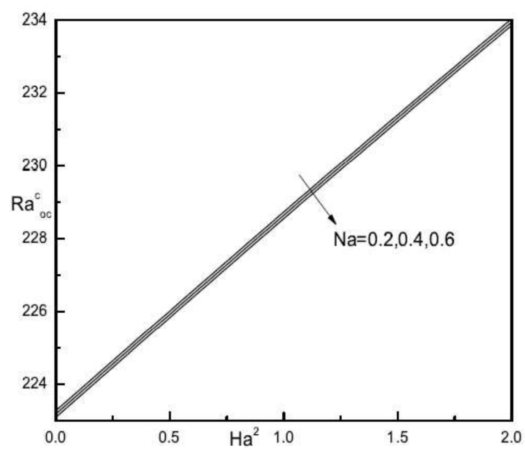

Figure 4: Change of $\boldsymbol{Ra^c_{oc}}$ with Ha^2 for the fixed values of Pr = 5, $\boldsymbol{\lambda_1} = 0.4, \lambda_2 = 0.5, and\ Rn = 2.$

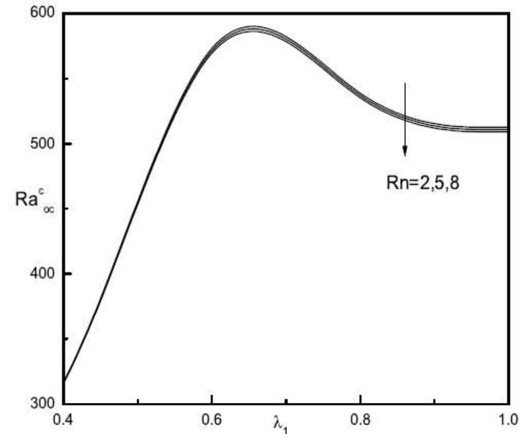

Figure 5: Change of $\boldsymbol{Ra_{oc}^{c}}$ with λ_1 for the fixed values of Pr = 5, Ha^2 = 2, λ_2 = 0.5, and Na = 0.5

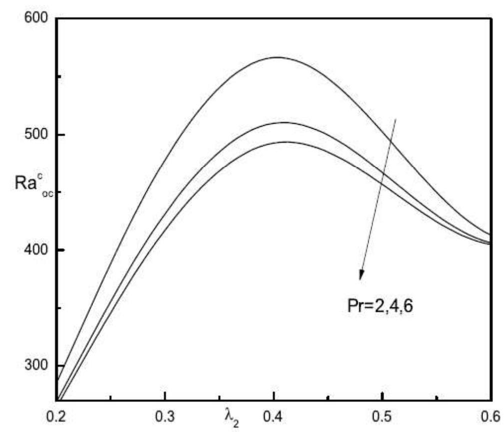

Figure 6: Change of $\boldsymbol{Ra_{oc}^{c}}$ with λ_1 for the fixed values of Ha^2= 5, Rn = 5, λ_1 = 0.5, and Na = 0.5

$$\left(1+\lambda_1\frac{\partial}{\partial t}\right)\left(\frac{1}{Pr}\left(\frac{\partial V}{\partial t}+(\mathrm{V}.\nabla)\mathrm{V}\right)-\nabla\mathrm{P}+R_m\hat{e}_z-RaT\hat{e}_z+R_n\phi\hat{e}_z-Ha^2[(\mathrm{V}\times\hat{e}_z)\times\hat{e}_z]\right)=\left(1+\lambda_2\frac{\partial}{\partial t}\right)\nabla^2\mathrm{V}, \tag{29}$$

$$\left(\frac{\partial}{\partial t}+(V.\nabla)\right)\phi=\frac{N_A}{Le}\nabla^2T+\frac{1}{Le}\nabla^2\phi, \tag{30}$$

$$\left(\frac{\partial}{\partial t} + (V.\nabla)\right)T = w + \nabla^2 T + \frac{N_B}{Le}(\nabla T\,.\nabla\phi) + \frac{N_A N_B}{Le}(\nabla T\,.\nabla T), \quad (31)$$

To study the weakly nonlinear behaviour, we employ multiple scale analysis. The

governing equations, after eliminating *T* and ϕ, can be written as:

$$\mathcal{L}\,w = \mathcal{N} \quad (32)$$

Where,

$$\begin{cases} \mathcal{L} = A_{11}B_{11}C_{11}\nabla^2 - \left(1+\lambda_1\frac{\partial}{\partial t}\right)Ha^2B_{11}C_{11}D^2 + \left(1+\lambda_1\frac{\partial}{\partial t}\right)\nabla_h^2 RaC_{11}, \\ N = N_{11} - \left(1+\lambda_1\frac{\partial}{\partial t}\right)\nabla_h^2 RaN_{22} + \left(1+\lambda_1\frac{\partial}{\partial t}\right)\nabla_h^2 RnB_{11}N_{33} + \left(1+\lambda_1\frac{\partial}{\partial t}\right)\nabla_h^2 RnB_{11}\,N_{33} + \left(1+\lambda_1\frac{\partial}{\partial t}\right)\nabla_h^2 Rn\frac{N_A}{Le}\nabla^2 N_{22}, \\ A_{11} = \left(1+\lambda_2\frac{\partial}{\partial t}\right)\nabla^2 - \left(1+\lambda_1\frac{\partial}{\partial t}\right)\frac{1}{Pr}\frac{\partial}{\partial t}, \\ B_{11} = \frac{\partial}{\partial t} - \nabla^2, C_{11} = \frac{\partial}{\partial t} - \frac{1}{Le}\nabla^2, \\ N_{11} = \frac{-1}{Pr}\left(1+\lambda_1\frac{\partial}{\partial t}\right)\nabla\times\nabla\times(V.\nabla)V.\hat{e}_z, \\ N_{22} = -(V.\nabla)T + \nabla^2 T + \frac{N_B}{Le}(\nabla T.\nabla\phi) + \frac{N_A N_B}{Le}(\nabla T.\nabla T), \\ N_{33} = -(V.\nabla)\phi \end{cases}$$

We write u, v, w, T and ϕ as,

$$\boldsymbol{u} = \boldsymbol{\epsilon u_0} + \boldsymbol{\epsilon^2 u_1} + \boldsymbol{\epsilon^3 u_2} + \cdots,$$

$$\boldsymbol{v} = \boldsymbol{\epsilon v_0} + \boldsymbol{\epsilon^2 v_1} + \boldsymbol{\epsilon^3 v_2} + \cdots,$$

$$\boldsymbol{w} = \boldsymbol{\epsilon w_0} + \boldsymbol{\epsilon^2 w_1} + \boldsymbol{\epsilon^3 w_2} + \cdots,$$

$$\boldsymbol{T} = \boldsymbol{\epsilon T_0} + \boldsymbol{\epsilon^2 T_1} + \boldsymbol{\epsilon^3 T_2} + \cdots,$$

$$\boldsymbol{\phi} = \boldsymbol{\epsilon \phi_0} + \boldsymbol{\epsilon^2 \phi_1} + \boldsymbol{\epsilon^3 \phi_2} + \cdots,$$

(33)

Where,

$$\epsilon^2 = \frac{Ra - Ra_{sc}}{Ra_{sc}} \ll 1$$

The first approximations are,

$$u_0 = \frac{i\Pi}{q_{sc}}\left[Ae^{iq_{sc}x}cos\Pi z - c.c\right]$$
$$w_0 = \left[Ae^{iq_{sc}x}\sin\Pi z + c.c\right],$$
$$T_0 = \frac{1}{\delta^2}\left[Ae^{iq_{sc}x}\sin\Pi z + c.c\right],$$
$$\phi_0 = \frac{-Na}{\delta^2}\left[Ae^{iq_{sc}x}\sin\Pi z + c.c\right]$$
(34)

Where,

$$\begin{cases} A = A(X,Y,Z,T) - amplitude, \\ \quad c.c. - complex\ conjugate \end{cases}$$

In our analysis, we proceed to scale the slow variables (X, Y, Z, and T) in the following manner,

$$X = \epsilon x, \qquad Y = \epsilon^{\frac{1}{2}}y, \qquad Z = z, \qquad T = \epsilon^2 t,$$

Based on these scalings, the differential operators are expressed as:

$$\frac{\partial}{\partial x} \longrightarrow \frac{\partial}{\partial x} + \epsilon\frac{\partial}{\partial X},$$
$$\frac{\partial}{\partial y} \longrightarrow \frac{\partial}{\partial y} + \epsilon^{\frac{1}{2}}\frac{\partial}{\partial Y},$$
$$\frac{\partial}{\partial z} \longrightarrow \frac{\partial}{\partial Z},$$
$$\frac{\partial}{\partial t} \longrightarrow \epsilon^2\frac{\partial}{\partial T}. \qquad (35)$$

By using Eq. (35), the operators $\mathcal{L}$ and $\mathcal{N}$ of Eq. (32) can be written as,

$$\mathcal{L} = \mathcal{L}_0 + \epsilon\mathcal{L}_1 + \epsilon^2\mathcal{L}_2 \ldots,$$

$$\mathcal{N} = \mathcal{N}_0 + \epsilon\mathcal{N}_1 + \epsilon^2\mathcal{N}_2 \ldots \quad (36)$$

On putting Eq. (36) into Eq. (32), one obtains the following system,

$$\mathcal{L}_0 w_0 = 0 \quad (37)$$

$$\mathcal{L}_0 w_1 + \mathcal{L}_1 w_0 = \mathcal{N}_0, \quad (38)$$

$$\mathcal{L}_0 w_2 + \mathcal{L}_1 w_1 + \mathcal{L}_2 w_0 = \mathcal{N}_1. \quad (39)$$

Where,

$$\mathcal{L}_0 = -\frac{1}{Le}\nabla_h^2 Ra\nabla^2 - \frac{1}{Le}\nabla_h^2 NaRn\nabla^2 - \frac{1}{Le}Ha^2\left(\frac{\partial}{\partial z}\right)^2(\nabla^2)^2 + \frac{1}{Le}(\nabla^2)^4,$$

$$\mathcal{L}_1 = 2\frac{\partial^2}{\partial x\partial X}\frac{1}{Le}\left(4\nabla^6 - \nabla_h^2(Ra + NaRn) - \nabla^2\left(Ra + NaRn + 2Ha^2\left(\frac{\partial}{\partial z}\right)^2\right)\right),$$

$$\mathcal{L}_2 = \frac{\partial}{\partial T}\frac{1}{LePr}\left(-\nabla^6(1 + Pr + LePr) + LePrRa\nabla_h^2 - \nabla^2 Ha^2 Pr\lambda_1\left(\frac{\partial}{\partial z}\right)^2\right) + \frac{\partial}{\partial T}\frac{1}{LePr}\left(\nabla^2 Pr\left((1 + Le)Ha^2\left(\frac{\partial}{\partial z}\right)^2 - \nabla_h^2\lambda_1(Ra + NaRn)\right) + \nabla^8 Pr\lambda_2\right) + \frac{\partial^2}{\partial X^2}\frac{1}{Le}\left((4\nabla^6 - \nabla_h^2 (Ra + NaRn) - \nabla^2\left(Ra + NaRn + 2Ha^2\left(\frac{\partial}{\partial z}\right)^2\right)\right) + \left(2\frac{\partial^2}{\partial x\partial X}\right)^2\frac{1}{Le}\left(6\nabla^4 - (Ra + NaRn) - Ha^2\left(\frac{\partial}{\partial z}\right)^2\right)$$

Let us substitute the solution $\mathcal{L}_0 w_0 = 0$ and one obtains,

$$Ra_{sc} = -RnN_a + Ha^2\pi^2\frac{\delta^2}{q^2} + \frac{\delta^6}{q^2}, \quad (40)$$

which is well agreed with stationary Rayleigh number. From the equation $\mathcal{L}_0 w_1 + \mathcal{L}_1 w_0 = \mathcal{N}_0$, $\mathcal{N}_0 = 0$ and $\mathcal{L}_0 w_0 = 0$. The equation reduces to,

$$w_1 = 0, \tag{41}$$

$$u_1 = 0, \tag{42}$$

$$T_1 = \frac{-1}{2\pi\delta^2}|A|^2 sin2\pi z. \tag{43}$$

$$\phi_1 = \frac{NaLe}{2\pi\delta^2}\left(\frac{1}{Le}+1\right)|A|^2 sin2\pi z \tag{44}$$

When these solutions are substituting into Equation (39), we derive the Newell-Whitehead equation in the following manner:

$$\zeta_0\frac{\partial A}{\partial T} - \zeta_1\left(\frac{\partial}{\partial X} - \frac{i}{2q}\frac{\partial^2}{\partial Y^2}\right)^2 A - \zeta_2 A + \zeta_3|A|^2 A = 0 \tag{45}$$

Where,

$$\zeta_0 = \frac{1}{LePr}(\delta^6(1+Pr+LePr) - LePrRaq^2 - Ha^2Pr\lambda_1\delta^2\pi^2) - \frac{1}{LePr}(\delta^2\mathrm{Pr}(q^2\lambda_1(Ra+NaRn) + (1+Le)Ha^2\pi^2) - Pr\lambda_2\delta^8), \tag{46}$$

$$\zeta_1 = \frac{1}{Le}(6\delta^4 - (Ra+NaRn) + Ha^2\pi^2), \tag{47}$$

$$\zeta_2 = Raq^2\delta^2, \tag{48}$$

$$\zeta_3 = -\frac{q^2}{2Le}[Ra+RnNa] \tag{49}$$

In order to find the maximum amplitude we consider only the x-dependence terms in Eq. (45),

$$\frac{d^2A}{dX^2} + \frac{\zeta_2}{\zeta_1}\left(1 - \frac{\zeta_3}{\zeta_1}|A|^2\right)A = 0 \tag{50}$$

$$\therefore A(X) = A_0\tanh\left(\frac{X}{\wedge_0}\right) \tag{51}$$

Where,

$$\begin{cases} A_0 = \left(\dfrac{\zeta_2}{\zeta_3}\right)^{\frac{1}{2}}, \\ \wedge_0 = \left(\dfrac{2\zeta_1}{\zeta_2}\right)^{\frac{1}{2}} \end{cases}$$

5. Heat transport by convection

From Eq. (32), we obtain the maximum of steady amplitude ($|A_{max}|$) as,

$$|A_{max}| = \left(\frac{\epsilon^2 \zeta_2}{\zeta_3}\right)^{\frac{1}{2}}$$

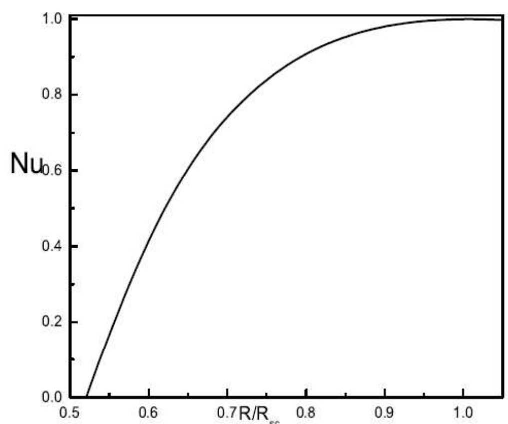

Figure 7: The figure is plotted for fixed values of *Le* = 2, *Pr* = 5, *Rn* = 0.2, *Na* = 2, λ_1 = 0.4, λ_2 = 0.5, and Ha^2 = 2.

Hence,

$$Nu = 1 + \frac{\epsilon^2}{\delta_{Sc}^2} |A_{max}|^2$$

is Nusselt number. From Eq. (37), we obtain convection for $R > R_{T_{Sc}}$ and conduction for $R \leq R_{T_{Sc}}$. From Eq. (32), is valid for $\zeta_3 > 0$ which is possible when $R > R_{T_{Sc}}$, this we get,

1) convection for $Nu > 1$,
2) convection for $Nu \geq 1$ (see in Figure 7).

6. Results and Conclusions

This section presents a discussion of the results. In this part, we evaluated an analytical study of magnetic effect on the onset of convection of a Oldroyd-B nanofluid. The critical Rayleigh number at the onset of stationary (Ra_{sc}^c) and oscillatory (Ra_{oc}^c) convection is obtained for the prescribed values of the physical parameters.

In Figs. 2-3, we show the change of critical Ra_{sc} with different physical parameters. Fig. 2 shows the relation between critical Ra_{sc} and Ha^2. The threshold value of Ra is an increasing function of Ha^2. A higher Hartmann number shows that strength in a magnetic field, which will enhance the stability of the fluid layer, leads to a delay in the convection, so a higher temperature difference (higher critical Ra_{sc}) is needed to trigger convective motion. Variation of Ra_{sc}^c with Na is shown in Fig. 3. It is evident that as Na increases, the critical Ra_{sc} also increases, indicating the stabilising effect. Furthermore, in Figs. 2-3, the destabilising nature of Rn on steady instability has been also observed.

Stationary Rayleigh number is independent of *Pr*, λ_1, *and* λ_2, which can be seen in Eq. 27. Hence, *Pr*, λ_1, and λ_2 do not effect the stationary instability.

In Figs. 4-6, variation of critical Ra_{oc} with different physical parameters has been displayed. The effect of Ha^2 and *Na* on critical Ra_{oc} is shown in Fig. 4. As Ha^2 increases critical Ra_{oc} increases whereas on increasing the parameter *Na*, critical Ra_{oc} decreases. Hence $Ha^2(Na)$ has stabilising (destabilising) effect on oscillatory convection.

Critical Ra_{oc} is monotonically decreasing function of *Pr* and *Rn*, which has been evident from Figs. 5-6. Hence, as increasing the value of Pr and Rn advances the onset of convection. Moreover, a non-monotonic trend of Ra_{oc}^c with λ_1, and λ_2. As increasing the value of λ_1, and λ_2, critical Ra_{oc} initially increases gradually later it decreases.

7. Conclusions

The problem of convective instability of a Oldroyd-B nanofluid with magnetic effect is considered in this paper. Both linear and weakly nonlinear analyses are studied. One term Gallerkin approach is used to study the linear theory whereas multiple scale analysis is used to study the weakly nonlinear theory. From the results we conclude that Pr, λ_1, and λ_2 does not shown any effect on stationary convection. Moreover, Rn and Na has destabilizing effect on system whereas Hartmann number and Prandtl number has stabilizing effect on the flow. Critical Ra_{oc} is non-monotonic function of λ_1, and λ_2. In weakly nonlinear analysis, a multiple scale approach is used to derive amplitude equation. Heat transport is studied by calculating Nusselt number using amplitude.

Conflict of Interest

The authors hereby declare no potential conflicts of interest with respect to the research, funding, authorship, and/or publication of this article

References

[1] Choi, S.: Enhancing thermal conductivity of fluids with nanoparticles. In: Siginer D.A., Wang, H.P. (eds.) Developments and Applications of Non-Newtonian Flows, ASME FED- Vol. 231/ MD-Vol. 66, NewYork, pp. 99105, 1995.

[2] Masuda, H., Ebata, A., Teramae, K., Hishinuma, N.: Alteration of thermal conductivity and viscosity of liquid by dispersing ultra-fine particles. Netsu Bussei 7, 227233,1993.

[3] Buongiorno, J., Hu, W.: Nanofluid coolants for advanced nuclear power plants, Paper no. 5705. In: Proceedings of ICAPP 05, Seoul, 2005.

[4] Tzou, D.Y.: Instability of nanofluids in natural convection. ASME J. Heat Transf. 130, 072401, 2008

[5] Tzou, D.Y.: Thermal instability of nanofluids in natural convection. Int. J. Heat Mass Transf. 51, 29672979, 2008.

[6] Nield, D.A., Kuznetsov, A.V.: The onset of convection in a nanofluid layer. ASME J. Heat Transf. 2009.

[7] Buongiorno, J.: Convective transport in nanofluids. ASME J. Heat Transf. 128, 240250, 2006.

[8] Nield, D.A., Kuznetsov, A.V.: Thermal instability in a porous layer saturated by a nanofluid. Int. J. Heat Mass Transf. 2009.

[9] S. Chakravarty, P.K. Mandal Unsteady flow of a two-layer blood stream past a tapered flexible artery under stenotic conditions, Computational Methods in Applied Mathematics, 4 (4), pp. 391-409,2004.

[10] K. Tzirakis, L. Botti, V. Vavourakis, Y. Papaharilaou Numerical modeling of nonnewtonian biomagnetic fluid flow Comput. Fluids, 126, pp. 170-180, 2016.

[11] D. Sankar, U. Lee Twofluid casson model for pulsatile blood flow through stenosed arteries: a theoretical model Commun. Nonlinear Sci. Numer. Simul., 15 (8) , pp. 2086-2097, 2010.

[12] J. Boyd, J.M. Buick, S. Green Analysis of the casson and carreau-yasuda nonnewtonian blood models in steady and oscillatory flows using the lattice Boltzmann method Phys. Fluids, 19 (9), pp. 093-103, 2007.

[13] Bugliarello G, Sevilla J. Velocity distribution and other characteristics of steady and pulsatile blood flow in fine glass tubes. Biorheology, 7:85107, 1970.

[14] Sharan M, Popel AS. A two-phase model for flow of blood in narrow tunes with increased effective viscosity near the wall. Biorheology, 28:41528, 2001.

[15] Chakravarthy S, Sarifuddin S, Mandal PK. Unsteady flow of a two-layer blood stream past a tapered flexible artery under stenotic conditions. Comput Meth Appl Math, 4:391409, 2004.

[16] Scott Blair GW. The success of Casson equation. Rheol Acta, 5:1847, 1966.

[17] Iida N. Influence of plasma layer on steady blood flow in micro-vessels. Jpn J Appl Phys, 17:20314, 1978.

[18] Dash RK, Jayaraman G, Metha KN. Estimation of increased flow resistance in a narrow catheterized artery a theoretical model. J Biomech, 29:91730, 1996.

[19] Srivastava VP. Two-phase model of blood flow through stenosed tubes in the presence of a peripheral layer: applications. J Biomech 1996;29:137782.

[20] Mandal PK. An unsteady analysis of non-linear two-layered 2D model of pulsatile flow through stenosed arteries. Math Model Anal 2003;8:22946.

[21] Chaturani P, Ponnalagar Samy R. Pulsatile flow of Cassons fluid through stenosed arteries with applications to blood flow. Biorheology, 23:499511, 1986.

[22] Reddy, G. S. K., Bhavyashree, S. M., Ragoju, R., & Yadav, D. (2025). Viscous dissipation effects in thermosolutal convection of Casson fluid in porous layer with vertical throughflow: Global stability analysis. Chinese Journal of Physics.

[23] Reddy, G., Ragoju, R., & Shekhar, S. (2023). Thermohaline convection of a Casson fluid in a porous layer: Linear and non-linear stability analyses. Physics of Fluids, 35(9).

[24] Gupta, U., Sharma, J., & Devi, M. (2021). Double-diffusive instability of Casson nanofluids with numerical investigations for blood-based fluid. The European Physical Journal Special Topics, 230(5), 1435-1445.

[25] Chandrasekhar, S. (2013). Hydrodynamic and hydromagnetic stability. Courier Corporation.