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On Equality and Strong Equality of Domination Number and Independent Domination Number in Graphs

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Abstract

In this paper we explore graphs having same domination number γ and independent domination number γ_i. Such graphs are denoted as (γ, γ_i)-graphs. Several families of (γ, γ_i)-graphs have been constructed. The realization problem for graphs with $\gamma = \gamma_i = a$ for any given positive integer a has been solved. Furthermore, properties of graphs in which every γ-set is a γ_i-set has been investigated.

Keywords: Domination, Independent domination, Efficient dominating set, Strong equality of domination parameters.

1. Introduction

Let graph G = (V, E) be a simple finite connected graph of order n. The *open neighborhood* of a vertex v $\in$ V is N(v) = {u $\in$ V | vu $\in$ E}. The *closed neighborhood* is N[v] = N(v) $\cup$ {v}. A *corona* $G \circ K_1$ of a graph G is the graph obtained from G by adding a new vertex u and

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the pendent edge uv for every vertex v $\in$ V(G). *Subdivision* is the operation of replacing an edge by a path of two edges through a new vertex. A graph obtained from a graph H by successive edge subdivisions is called *H-subdivision (or subdivision of H).*

A vertex of degree one is called a *leaf*. A *support vertex* is a vertex that is adjacent to a leaf, while a *strong support vertex* is adjacent to at least two leaves. A tree with at most two vertices of degree more than one is called a *double star*.

A set S of vertices is a *dominating set* if every vertex in V is either in S or adjacent to a vertex in S. The minimum cardinality of a dominating set is the *domination number* γ (G). A dominating set of cardinality γ (G) is called γ*-set* of G (or simply γ (G)-set) .

The fascinating fact is that we can define various types of domination parameters by imposing several graph theoretic properties in the definition of dominating set. A dominating set is an *independent dominating set* if it is independent. An *independent domination number* γ_i is the minimum cardinality of an independent dominating set. We call a minimum independent dominating set of a graph G a γ_i-set of G or γ_i(G)-set.

Since every independent dominating set is a dominating set, for any graph G, γ(G) $\leq \gamma_i$(G). There are graphs for which independent domination number is same as the domination number. However, it is not necessary that every γ-set will be a γ_i-set for a graph with γ = γ_i. First we explore graphs whose independent domination number is the same as its domination number.

2. (γ, γ_i) -Graphs

We call graphs G which satisfies the relation γ(G) = γ_i(G) as (γ, γ_i) -graphs. The complete graph K_n, star graph $K_{1,t}$, wheel graph W_n, path graph P_n and cycle graph C_n are some of the standard classes of (γ, γ_i) - graphs.

Theorem 1. If G is a graph with γ_i(G)$\leq$2, then G is a (γ, γ_i) -graph.

Proof. We have 1 $\leq$ γ (H) $\leq$ γ_i (H) for every graph H. It follows that if G is a graph with γ_i(G) = 1, then γ (G) = γ_i(G). Suppose G be a graph with γ_i(G) = 2. If possible, let γ(G) = 1,then since every singleton set is an independent set, γ_i(G) = 1, a contradiction. Therefore, if G is a graph with γ_i(G) = 2, then γ (G) = γ_i(G). Hence, if G is a graph with γ_i(G) $\leq$ 2, then G is a(γ, γ_i) -graph.

If a graph G has a set S such that the collection of closed neighborhoods of vertices of S gives a partition, then S is definitely a dominating set of G. Bange *et al.* [1] have named such a set S such that {N[s] : s $\in$S } is a partition as *Efficient Dominating Set(or EDS)*. Goddard and Henning [6] have shown that a graph having an EDS is a (γ, γ_i) -graphs. Note that it is not compulsory for a graph to have an EDS. For instance, there is no set X $\subseteq V(C_4)$ such that collection of closed neighborhoods of vertices of X forms a partition in cycle C_4.

If S is an EDS of a graph G, then the closed neighborhoods of vertices of S are disjoint sets. This implies that S is a set of non-adjacent vertices. Therefore, the following remark is quite obvious.

Remark 1. An efficient dominating set of a graph G is independent.

The converse of **Remark 1** is not true always. Every EDS is a dominating set, but keep in mind that every γ(G)- set need not be an EDS . We can observe that an independent set of C_4 is not an EDS and no $\gamma(C_4)$-set is an EDS.

2. Construction of families of (γ, γ_i) –graphs

In this section, we attempt to construct some families of graphs having equal domination number and independent domination number.

2.1.1 Dumbbell graphs $D_{r,s}$

Consider two complete graphs K_r and K_s on r and s vertices, respectively. Connect K_r and K_s by an edge e. Thus we obtain a simple graph called Dumbbell graph, denoted by $D_{r,s}$.

Proposition 1. Dumbbell graph $D_{r,s}$ is a (γ, γ_i) –graph.

Proof. We claim that $\gamma(D_{r,s}) = \gamma_i(D_{r,s})$. Let u $\in V(K_r)$ and v $\in V(K_s)$. Then the set {u, v} is a dominating set of $D_{r,s}$ for any vertices u and v. So, $\gamma(D_{r,s}) \leq 2$. Further, there is no vertex of degree (r + s - 1), showing that $\gamma(D_{r,s}) = 2$. Moreover, the set {u, v} is an independent dominating set for all non-adjacent pair of vertices u and v such that u $\in V(K_r)$ and v $\in V(K_s)$. Hence, $\gamma_i(D_{r,s}) \leq 2$. Since every independent dominating set is a dominating set we conclude that $\gamma_i(D_{r,s}) = 2$.

Figure 1 depicts Dumbbell graph $D_{3,4}$ obtained by connecting complete graphs K_3 and K_4 by an edge.

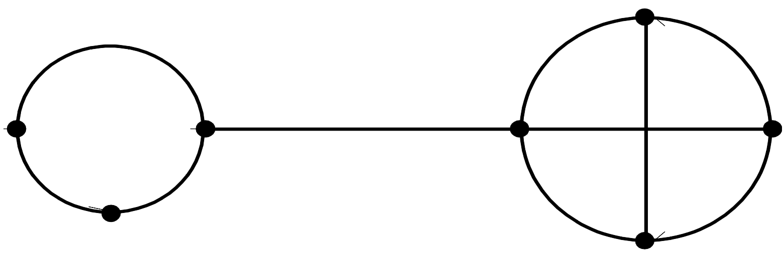

Fig 1. $\gamma(D_{3,4}) = \gamma_i(D_{3,4}) = 2$

Now we proceed to construct another family of (γ, γ_i) –graphs.

2.1.2 Sunrise graphs S_r

Consider a complete graph K_r and r- copies of star $K_{1,2}$. Connect a leaf of $K_{1,2}$ to each vertex of K_r using an edge. Thus we obtain a simple graph and we call it a Sunrise graph, denoted by S_r.

Figure 2 shows Sunrise graph S_3. It is obtained by connecting a leaf of $K_{1,2}$ to each vertex of complete graph K_3 by an edge.

Since no vertex of S_r has three pair-wise non-adjacent neighbors, sunrise graphs are claw-free. Therefore, S_r is a (γ, γ_i) –graph as every claw-free graphs are (γ, γ_i) –graphs [5, 6].

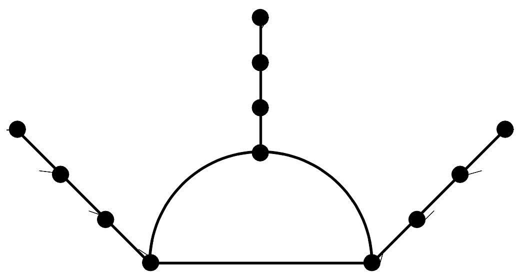

Fig 2. $\gamma(S_3) = \gamma_i(S_3) = 4$

Let z be a vertex of complete graph K_r and $x_1, x_2, \ldots, x_r$ be the central vertices of stars in sunrise graph S_r . Then $\{z, x_j \mid 1 \le j \le r\}$ is the $\gamma(S_r)$-set. In addition, these r+1 non-adjacent vertices are required to dominate the graph . Therefore, $\gamma(S_r) = \gamma_i(S_r)$ = r + 1. Hence we have the following proposition.

Proposition 2. Let S_r be a sunrise graph. Then $\gamma(S_r) = \gamma_i(S_r)$ = r + 1.

The above proposition also leads us to the realization problem of constructing a graph G such that γ(G) = γ_i(G) = k for any positive integer 'k'. This is achieved by considering the sunrise graph S_{k-1}.

Theorem 2. Given any positive integer ' k', there exists a graph G with γ (G) = γ_i(G) = k.

3. Strong equality between γ and γ_i

This section deals with the concept called strong equality of domination number and independent domination number in graphs. This concept of strong equality of two graph theoretic parameters was introduced by Haynes *et al.* [2, 4].

Let P_1 and P_2 be properties of vertex subsets of a graph G, and assume that every subset of V(G) with property P_2 also has property P_1. Let ψ_1(G) and ψ_2(G) denote the minimum cardinalities of sets with properties P_1 and P_2 respectively. Then ψ_1(G) $\leq \psi_2$(G). If ψ_1 (G) = ψ_2 (G) and every ψ_1 (G)-set is also a ψ_2 (G)-set, then we say that ψ_1 (G) *strongly equals* ψ_2 (G), written $\psi_1(G) \equiv \psi_2(G)$. It is shown that for a path P_n and cycle C_n

$$\gamma_i(C_{3k}) \equiv \gamma(C_{3k}) = \gamma_i(P_{3k}) \equiv \gamma(P_{3k}) = k$$
$$\gamma_i(C_{3k+2}) \equiv \gamma(C_{3k+2}) = \gamma_i(P_{3k+2}) \equiv \gamma(P_{3k+2}) = k+1$$

Note that γ and γ_i are not strongly equal for path P_{3k+1} and cycl C_{3k+1} even though $\gamma_i(C_{3k+1}) = \gamma(C_{3k+1}) = \gamma_i(P_{3k+1}) = \gamma(P_{3k+1}) = k+1$, where k is any positive integer [2].

If a graph G satisfies the relation $\gamma = \gamma_i$ and every γ-set of G is also a $\gamma_i(G)$-set, then $\gamma(G)$ strongly equals $\gamma_i(G)$, denoted as $\gamma(G) \equiv \gamma_i(G)$. We know that γ (P_5) = $\gamma_i(P_5)$ = 2 and each of the three γ (P5)-sets is also a γ_i-set of P_5. Therefore γ (P_5) $\equiv \gamma_i(P_5)$ .Whereas in cycle C_4 which also is a (γ, γ_i) –graph, we can see that not every γ-set is a γ_i-set and so $\gamma(C_4)$ and $\gamma_i(C_4)$ are not strongly equal.

3.1 Construction of families admitting strong equality between γ and γ_i

It is to be noted that γ and γ_i are strongly equal for path P_n and cycle C_n if n = 3k and n = 3k + 2 [2, 3, 4].

For the tree in **Figure 3**, the sets $\{z_i \mid 1 \le i \le t\} \cup \{w\}$ and $\{y_i \mid 1 \le i \le t\} \cup \{w\}$ are the γ sets and moreover, they are independent sets. Thus $\gamma_i = \gamma = t + 1$. Since every γ-set is a γ_i-set, γ strongly equals γ_i.

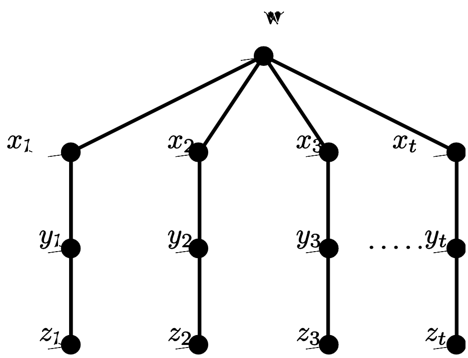

Fig 3. $\gamma_i \equiv \gamma = t + 1$

The set $\{b_i, p_i \mid 1 \le i \le t\}$ is the γ -set of the tree in **Figure 4**. As the same set is independent too, its domination number and independent domination number is equal to 2t . Therefore, it admits strong equality between γ and γ_i.

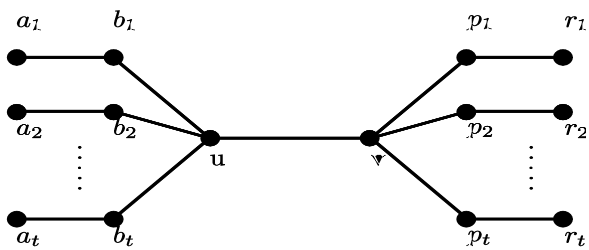

Fig 4. $\gamma_i \equiv \gamma = 2t$

Recall that paths and cycles are claw-free. But the path P_{3k+1} and cycle C_{3k+1} never admits strong equality between γ and γ_i .

The following observations provide few families of graphs which never possess strong equality.

Observation 1. Every claw-free graphs need not have strong equality between γ and γ_i.

Observation 2. The corona of graphs never admit strong equality between domination number and independent domination number.

Our aim is to construct a family of graphs admitting strong equality between γ and γ_i . For the purpose of construction, consider a complete graph K_q on q vertices and 'q' number of stars $K_{1,(q-1)}$. Then, connect an edge from the central vertex of a star to a vertex of K_q. The resulting graph is a simple graph and we wish to call it as **Umbel graph**, U_q as it bears some resemblance of an inflorescence called Umbel.

The **Figure 5** shows Umbel graph U_4

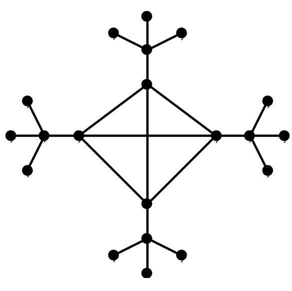

Fig 5. $\gamma(U_4) = \gamma_i(U_4) = 4$

This family of graphs can be extended to a wider class. Consider a complete graph K_q on q vertices and $|V(K_q)|$ number of stars on vertices $t_1, t_2, ..., t_q$ where t_i, $1 \le i \le q$ is an integer. Next, connect an edge from the center vertex of a star to a vertex of K_q. We can observe that the family of resulting graphs is an extension of family of umbel graphs. These graphs admit strong equality between γ and γ_i as the only γ-set is the set of central vertices of all stars and moreover, it is an independent set.

Hence, we have the following theorem.

Theorem 3. Given any positive integer q, there exists a graph G such that $\gamma(G) \equiv \gamma_i(G) = q$.

4. Conclusion

In this paper we have discussed several families of (γ, γ_i)-graphs. New families of such graphs have been constructed. We analyzed and discussed the concept of strong equality of domination parameters. Particularly, we focused on strong equality between γ and γ_i. New families of graphs having strong equality between γ and γ_i have been constructed. We proved that there exists a graph such that $\gamma(G) \equiv \gamma_i(G) = q$.

Studies could be extended to obtain more families of graphs with $\gamma \equiv \gamma_i$ and characterization of such graphs.

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