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Degree-Eccentricity Matrix of Graphs and Some Properties

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Abstract

This paper presents a new matrix for a given graph called the Degree-Eccentricity (DE) matrix, which consists of the degree and eccentricity of a vertex. Properties such as irreducibility and primitivity of this matrix are discussed. Further we obtain the spectrum and energy of DE matrices associated with various classes of graphs and some graphs obtained through graph operations. Also, we try to develop an algorithm to construct a new class of graph with DE energy equal to one. Further, we made an attempt to discover few graphs with DE energy equal to one. Also, an upper bound for the eigenvalues of DE matrix is obtained.

Keywords: Degree-Eccentricity matrix, irreducibility, primitivity, Degree-Eccentricity energy.

AMS Subject Classification (2020): 05C07, 05C12, 05C76, 05C92.

1. Introduction

In this paper, all graphs which we consider are simple, connected and undirected. For a graph $G = (V(G), E(G))$ with number of vertices $n = |V(G)|$ and number of edges $m = |E(G)|$, the degree of a vertex v_i in G is defined as the number of edges incident on v_i and is denoted by $d(v_i)$ or simply d_i. For any two vertices u and v in G, the distance between these vertices is the length of the shortest path joining them and it is denoted by $d(u, v)$ in G. The eccentricity of v_i denoted by $e(v_i)$ or simply e_i is defined as $e(v_i) = max\{d(u, v_i)$: u is a vertex of $G\}$. A vertex u is a neighbour of v in G, if uv is an edge of G, and $u \neq v$. The set of all neighbours of v is the open neighbourhood of v or the neighbour set of v, and is denoted by $N(v)$; the set $N[v] = N(v) \cup \{v\}$ is the closed neighborhood of v in G [10]. G^n is the n^{th} power of a graph G and it has the same vertex set as that of G and an edge adjacency between two vertices

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u and v in G^n is given whenever $d(u, v) \leq n$ in G. $G_1 \vee G_2$ denotes the join of the two graph G_1 and G_2 and has vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$ together with all edges connecting the vertices of the first vertex set to that of the second. For additional notations and terminologies not covered here, see the following [6].

The theory of matrices have huge applications in the domain of graph theory. When these two disciplines mixed together, we got an interesting area in graph theory namely spectral graph theory. Any finite graph can be represented in the form of a matrix in which the order and entries of the matrix depends on the respective graph. Adjacency, incidence and Laplacian matrices are a few among them.

After 1978, the concept 'Graph Energy' was presented to the mathematico-chemical community [3]. The total π-electron energy of conjugated hydrocarbon molecules is a chemical quantity that is strongly related to this graph invariant [12]. The introduction of this notion resulted in the discovery of various novel outputs, some of which have chemical relevance too. By considering various graph parameters, graph theorists have carried out a variety of studies on graph energies. Vertex energy, maximum degree energy, degree sum energy, eccentricity energy, eccentricity extended energy etc are different energies which are associated with the vertex degree and eccentricity of a graph. A survey of graph energies is given in [2]. Despite being developed only for mathematical research, graph energy and its later variations have fascinating, rather unexpected, and enigmatic uses in other scientific and practical domains. There are many clear-cut uses of the graph energy in the field of chemistry, particularly associated with unsaturated conjugated compounds, which are not covered in detail here. Application in crystallography, macromolecule theory, and protein sequence analysis and comparison are all somewhat linked. Attempts to use graph energies in network analysis, such as in air transportation, satellite communication, and biology, are also very interesting [8].

Motivated by the relevance of this area and tremendous applications, we are studying a new type of matrix and energy associated with the matrix called Degree-Eccentricity 2 (DE) matrix. The DE energy is defined in the same manner with the ordinary graph energy. The DE energy of a given graph G can be defined as the sum of the absolute values of the eigenvalues of the corresponding DE matrix of G. The DE matrix is not symmetric, whereas the adjacency matrix and other matrices of a connected graph are symmetric in nature.

2. Preliminaries

Definition 2.1. *[6] The* **degree sequence** *of a graph G is the degree of vertices of the graph G arranged in non-increasing order.*

Definition 2.2. *[11] Two graphs with the same degree sequence are said to be* **degree equivalent**.

Lemma 2.1. *[6] For any simple graph G with order $n \geq 2$, G has at least two vertices of the same degree.*

Definition 2.3. [10] *For the graph G with vertices* $v_1, v_2, \ldots, v_n$, *the* **adjacency matrix** *of G (with the given labeling of the vertices $v_1, v_2, \ldots, v_n$) is an $n \times n$ matrix $A = (a_{ij})$, where*

$$a_{ij} = \begin{cases} 1, & if\ v_i\ and\ v_j\ are\ adjacent\ in\ G \\ 0, & otherwise. \end{cases}$$

Definition 2.4. [10] *Let the eigenvalues of* G, $\lambda_1, \lambda_2, \ldots, \lambda_n$ *be arranged in their non-decreasing order given by $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. For the distinct eigenvalues of G be $\lambda_1, \lambda_2, \ldots, \lambda_s$ with multiplicity m_i is the multiplicity of λ_i as an eigenvalues of G, we* write

$$S_P\,(G) = \begin{pmatrix} \lambda_1 & \lambda_2 \ldots\ldots \lambda_s \\ m_1 & m_2 \ldots m_s \end{pmatrix}$$

The spectrum of the adjacency matrix of A G is called the spectrum of the graph G.

Definition 2.5. *[10] The* **energy** *of G is defined as the sum of the absolute values of eigenvalues the graph G.*

Hence, the energy of the graph *G*, *ε(G)* of order *n* with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ is given by

$$\varepsilon(G) = \sum_{i=1}^{n} |\lambda_i|$$

Definition 2.6. *[13] Let $A \in M_n$ and λi, $i = 1, 2, \ldots, n$ be the eigenvalues of A. The* **spectral radius** *of A, denoted by ρ(A), is defined as*

$\rho(A) = max\ \{|\lambda_i| : i = 1, 2, \ldots, n\}$,

where M_n is the set of real matrices of order $n \times n$.

Definition 2.7. *[9] The matrix A is said to be similar to a matrix B (written $A \sim B$) if and only if there is a matrix P such that $B = PAP^{-1}$.*

Definition 2.8. *[13] If there is a permutation matrix P such that $P^TXP = Y$, then the two matrices X and Y are said to be* **permutation similar**.

Definition 2.9. *[13] If A is permutation similar to a matrix of the form*

$$\begin{bmatrix} B & 0 \\ C & D \end{bmatrix}$$

then the matrix $A \in M_n$ is said to be **reducible**, *where B and D are square matrices. A matrix which is not reducible is termed as* **irreducible**.

For a matrix of order 1, by definition it is irreducible. Again it is by definition that a square matrix with a zero row or a zero column is reducible.

Definition 2.10. [13] *Let A be a square irreducible non-negative matrix. Suppose A has exactly k eigenvalues of modulus ρ(A). The number k is the* **index of imprimitivity** *of A. If k = 1, then A is said to be* **primitive**; *otherwise A is considered to be imprimitive.*

3. The Degree-Eccentricity Matrix of a graph

Cauchy's matrix is a well-known concept in Linear algebra [5]. They are served as the basic components in decomposition formulas and fast algorithms for numerous displacement-structured matrices [4]. The Degree-Eccentricity matrix associated with a graph is a new matrix in which the above matrix served as its motivation. Using the two graph theoretical parameters viz. vertex degree and eccentricity, we define, Degree-Eccentricity matrix of a graph *G* as follows.

Definition 3.1. The **Degree-Eccentricity** matrix *of a graph G having degree sequence (d_1, d_2, . . . , d_n) is the square matrix DE(G) = [a_{ij}] of order n ≥ 2, in which* a_{ij} *is defined as*

$$a_{ij} = \frac{1}{d_i + e_j}, i, j = 1,2, \dots, n,$$

where $d_i = d(v_i)$ and $e_i = e(v_i)$.

An illustration is given below.

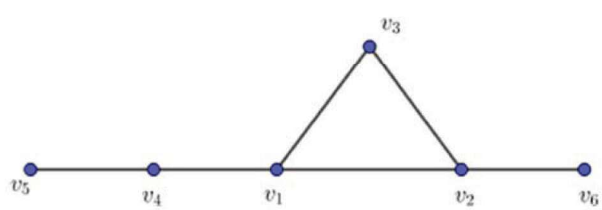

Figure 1: Graph G with degree sequence (3, 3, 2, 2, 1, 1)

(3, 3, 2, 2, 1, 1) is the degree sequence of *G* and the corresponding sequence of eccentricity is (2, 3, 3, 3, 4, 4). The DE matrix of *G* is given by

$$DE(G) = \begin{bmatrix} \frac{1}{d_1+e_1} & \frac{1}{d_1+e_2} & \frac{1}{d_1+e_3} & \frac{1}{d_1+e_4} & \frac{1}{d_1+e_5} & \frac{1}{d_1+e_6} \\ \frac{1}{d_2+e_1} & \frac{1}{d_2+e_2} & \frac{1}{d_2+e_3} & \frac{1}{d_2+e_4} & \frac{1}{d_2+e_5} & \frac{1}{d_2+e_6} \\ \frac{1}{d_3+e_1} & \frac{1}{d_3+e_2} & \frac{1}{d_3+e_3} & \frac{1}{d_3+e_4} & \frac{1}{d_3+e_5} & \frac{1}{d_3+e_6} \\ \frac{1}{d_4+e_1} & \frac{1}{d_4+e_2} & \frac{1}{d_4+e_3} & \frac{1}{d_4+e_4} & \frac{1}{d_4+e_5} & \frac{1}{d_4+e_6} \\ \frac{1}{d_5+e_1} & \frac{1}{d_5+e_2} & \frac{1}{d_5+e_3} & \frac{1}{d_5+e_4} & \frac{1}{d_5+e_5} & \frac{1}{d_5+e_6} \\ \frac{1}{d_6+e_1} & \frac{1}{d_6+e_2} & \frac{1}{d_6+e_3} & \frac{1}{d_6+e_4} & \frac{1}{d_6+e_5} & \frac{1}{d_6+e_6} \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{7} & \frac{1}{7} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{7} & \frac{1}{7} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{5} & \frac{1}{5} \end{bmatrix} \quad (1)$$

Even though we interchange the vertices of the same degree as well as their eccentricities in the degree sequence of G, this labelling of vertices ensures that the corresponding matrices are similar. This has no effect on the graph's spectral properties because the spectrum of a matrix is a similarity invariant [9].

For the above graph G, the vertex v_1 and v_2 have same degree. The DE matrix obtained by interchanging the position of v_1 and v_2 in the degree sequence along with their corresponding eccentricities is given below

$$DE(G) = \begin{bmatrix} \frac{1}{6} & \frac{1}{5} & \frac{1}{6} & \frac{1}{6} & \frac{1}{7} & \frac{1}{7} \\ \frac{1}{6} & \frac{1}{5} & \frac{1}{6} & \frac{1}{6} & \frac{1}{7} & \frac{1}{7} \\ \frac{1}{5} & \frac{1}{4} & \frac{1}{5} & \frac{1}{5} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{5} & \frac{1}{4} & \frac{1}{5} & \frac{1}{5} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{4} & \frac{1}{4} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{3} & \frac{1}{4} & \frac{1}{4} & \frac{1}{5} & \frac{1}{5} \end{bmatrix}$$

For convenience, the matrix in Equation (1) is taken as A and that in Equation (2) as B. Then we can find a matrix P with $B = PAP^{-1}$. In this case, the matrix P is given as

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

This shows that A and B are similar matrices.

The following is an immediate result of Lemma 2.1.

Remark 3.1. *For a graph G with n vertices,* $n \geq 2$, $\det(DE(G)) = 0$.

Remark 3.2. *Two-degree equivalent graphs need not have similar DE matrix.*

Consider the following example.

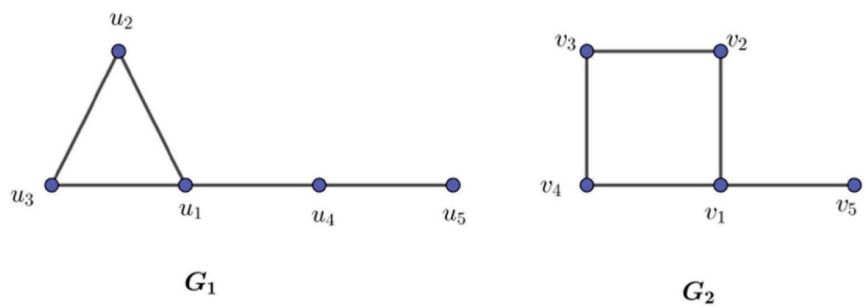

Figure 2: Degree equivalent graphs with different DE matrices

Here G_1 and G_2 have the same degree sequence as (3, 2, 2, 2, 1), but $DE(G_1)$ and $DE(G_2)$ are not similar.

$$DE(G_1) = \begin{bmatrix} \frac{1}{5} & \frac{1}{6} & \frac{1}{6} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{5} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{5} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{5} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{4} & \frac{1}{3} & \frac{1}{4} \end{bmatrix} \qquad DE(G_2) = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{6} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{5} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{5} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{5} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{4} & \frac{1}{3} & \frac{1}{4} \end{bmatrix}$$

4. Irreducibility and Primitivity of DE Matrix

The irreducibility and primitivity of DE matrices are the two characteristics to be focused in this section. Let A be a matrix which is said to be non-negative if all of its entries are non-negative real numbers. This implies that A is a positive matrix. We use notation $A > 0$. That is, A is a matrix with all of its entries are positive real numbers [13].

It is clear from the definition 3.1, all entries of a DE matrix associated with a graph on n vertices is greater than 0. That is, $a_{ij} > 0 \ \forall \ i, j = 1, 2, \ldots, n$. So we can say that $DE(G)$ is a positive matrix.

Lemma 4.1. [13] *A non-negative matrix A of order* $n \geq 2$ *is said to be irreducible if and only if* $(I + A)^{n-1} > 0$.

In the light of this Lemma, we can state the following.

Theorem 4.1. *Let G be a graph on n vertices, n ≥ 2. Then DE(G) is always irreducible.*

Next we state a known theorem.

Theorem 4.2. *[13] (Frobenius). Let A be an irreducible non-negative matrix. If the characteristic polynomial of A is*

$$\lambda^n + a_1\lambda^{n_1} + a_2\lambda^{n_2} + \cdots .. + a_t\lambda^{n_t},$$

where $n > n_1 > n_2 > \cdot\cdot\cdot > n_t$ and every $a_j \neq 0$, $j = 1, 2, \ldots, t$. Then the index of imprimitivity of A is given by

$$\gcd(n - n_1, n_1 - n_2, \ldots, n_{t-1} - n_t)$$

Theorem 4.3. *Let G be a graph with n ≥ 2 vertices. Then DE(G) is primitive.*

Proof. Since the determinant of *DE(G)* is 0, 0 must be an eigenvalue of the matrix.

Let *P(G; λ)* be the characteristic polynomial of *DE(G)*. Then it is of the form,

$$P(G;\lambda) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \cdots . + c_k\lambda^{n-k}$$

where k is the rank of DE(G).

Since DE(G) is a positive matrix, trace of DE(G) is a positive real number. This in turn implies that c1 ≠ 0 in the above equation. By Theorem 4.2, the index of imprimitivity of DE(G) is given by,

$$\gcd(n-(n-1), n-1 - (n-2), \ldots) = 1$$

The proof is over.

5. DE-Spectrum and Energy of Some Standard Graphs and New Graphs Obtained by Graph Operations

In this section we try to determine the DE-spectrum of some standard graphs and its DE energy. For a graph *G*, DE-spectrum of *G* is denoted by $DE\ S_p(G)$ and DE energy is denoted by $DE\ \varepsilon(G)$.

5.1. DE-Spectrum and Energy of C_n, n ≥ 3

In this section we determine the DE-spectrum of odd and even cycles.

Case 1: ***n*** **odd**

Since each vertex of Cn has degree 2 and eccentricity $\frac{n-1}{2}$, *n≥3, its DE matrix is given by*

$$DE(C_n) = \frac{2}{n+3}J_n$$

where J_n is the $n \times n$ matrix with all the elements are equal to 1. Clearly, rank of $DE(C_n)$ is 1. Hence, its DE-spectrum is given by

$$DE\ S_P(C_n) = \begin{pmatrix} \frac{2n}{n+3} & 0 \\ 1 & n-1 \end{pmatrix}$$

Case 2: n **even**

As in case 1, each vertex of C_n has degree 2 and eccentricity $\frac{n}{2}, n \geq 4$. The corresponding DE matrix is given by

$$DE(C_n) = \frac{2}{n+4} J_n.$$

Clearly, rank of DE(C_n) is 1. Hence, the DE-spectrum is given by

$$DE\ S_P(C_n) = \begin{pmatrix} \frac{2n}{n+4} & 0 \\ 1 & n-1 \end{pmatrix}$$

From the above two cases, the following proposition is put forth..

Proposition 5.1. *For the cycle* C_n, $n \geq 3$

$$DE\ \varepsilon(C_n) = \begin{cases} \frac{2n}{n+3}, & if\ n\ is\ odd \\ \frac{2n}{n+4}, & if\ n\ is\ even \end{cases}$$

5.2 DE-Spectrum and Energy of K_n, $n \geq 2$

It is clear that K_n is an $(n-1)$-regular graph with eccentricity of each vertex as 1. Its DE matrix is given by

$$DE(K_n) = \frac{1}{n} J_n$$

Rank of DE matrix of K_n is 1. Hence the DE-spectrum of K_n is given as follows.

$$DE\ S_P(K_n) = \begin{pmatrix} 1 & 0 \\ 1 & n-1 \end{pmatrix}$$

Hence, the following proposition holds.

Proposition 5.2. *For the complete graph* K_n, $n \geq 2$,

$$DE\ \varepsilon(K_n) = 1.$$

Next we try to compute the DE energy of the k^{th} power of a graph G, where $k = diam(G)$.

Corollary 5.1. *Let G be a graph with n vertices and let k = diam(G), then the DE* $\varepsilon(G^k) = 1$.

Proof. Let the vertex set of G be $\{v_1, v_2, \ldots, v_n\}$. Let the degree sequence be $(d_1, d_2, \ldots, d_n)$ and $(e_1, e_2, \ldots, e_n)$ be the corresponding sequence of eccentricities of the vertices. By definition of $diam(G)$, it is the $max\{e(v):v\in V(G)\}$. By taking $k = diam(G)$ and computing G^k, every pair of vertices whose distance $\leq k$ are adjacent. Since k is maximum eccentricity, so G^k results a complete graph on n vertices. By Proposition 5.2, $DE\ \varepsilon(G^k) = 1$.

An illustration of the above proposition is given below.

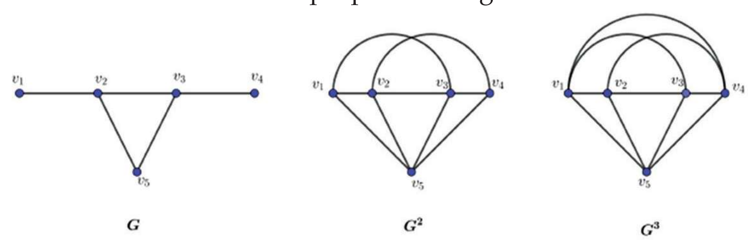

Figure 3: G and its powers

Here k = 3. Taking the 3rd power of G, it becomes K_4.

5.3. DE-Spectrum and Energy of Petersen graph

Since the Petersen graph G is 3 regular with eccentricity 2, we get the DE matrix as

$$DE(G) = \frac{1}{5}J_{10}$$

The corresponding DE-spectrum is given by

$$DE\ S_P(G) = \begin{pmatrix} 2 & 0 \\ 1 & 9 \end{pmatrix}$$

The DE energy is given by $DE\ \varepsilon\ (G) = 2$.

5.4. DE-Spectrum and Energy of $K_{m,n}$

Let the two partite sets of $K_{m,n}$ be V_1 and V_2, where $m \leq n$; $m, n \geq 2$. Then $|V_1| = m$, $|V_2| = n$. Let $\{u_1, u_2, \ldots, u_m\}$ be the vertices in V_1 and $\{v_1, v_2, \ldots, v_n\}$ be the vertices of V_2. Clearly, degree of each vertex in V_1 is n and that of V_2 is m. Also, the eccentricity of all vertices is 2. Then, the degree sequence of Km,n is given by

$$\left(\overbrace{n, n, \ldots, n}^{m\ times}, \overbrace{m, m, \ldots, m}^{n\ times}\right)$$

and the corresponding sequence of eccentricity is given by (2, 2, . . . , 2), where 2 is repeating $m + n$ times.

It follows that $DE\ (K_{m,n})$ is an $(m+n)\times(m+n)$ matrix of rank 1 and is given by

$$DE(K_{m,n}) = \begin{bmatrix} \frac{1}{n+2} J_{m\times(m+n)} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{1}{m+2} J_{n\times(m+n)} \end{bmatrix}$$

The DE-spectrum of $K_{m,n}$ is given by

$$DE\ S_p(K_{m,n}) = \begin{bmatrix} \frac{m^2+n^2+2(m+n)}{(m+2)(n+2)} & 0 \\ 1 & m+n-1 \end{bmatrix}$$

This leads to the following proposition.

Proposition 5.3. *For the complete bipartite graph $K_{m,n}$, $m,n \geq 2$; $m \leq n$,*

$$DE\varepsilon(K_{m,n}) = \frac{m^2+n^2+2(m+n)}{(m+2)(n+2)}$$

5.5. DE-Spectrum and Energy of Double graph of Certain Graphs

In this section we derive the DE-spectrum of double graph of cycle, star and friendship graph.

Definition 5.1. [7] *Let the vertex set of G be $V(G) = \{v_1, v_2, \ldots\ldots, v_p\}$. Take another copy of G with the vertex labels $\{u_1, u_2, \ldots, u_p\}$ where u_i corresponds to v_i for each i. Make u_i adjacent to all the vertices in $N(v_i)$ in G, for each i. The graph obtained in such a manner is called the* double graph *of G, and it is denoted by D_2G.*

5.5.1. DE-Spectrum and Energy of Double graph of Cycle, C_n, $n\geq 3$

Label the vertices of C_n by $v_1, v_2, \ldots, v_n$. In order to construct the double graph of C_n take a copy of C_n and label the vertices as $u_1, u_2, \ldots, u_n$, where each u_i corresponds to v_i, $i = 1, 2, \ldots, n$. Since v_i is adjacent to $v_{(i+1) \bmod n}$ and $v_{(i-1) \bmod n}$, where $i = 1, 2, \ldots, n$, join u_i with $v_{(i+1) \bmod n}$ and $v_{(i-1) \bmod n}$.

The resulting graph D_2C_n has $2n$ vertices, each vertex is of degree 4 and the eccentricity of each vertex is $\frac{n}{2}$, when n is even and $\frac{n-1}{2}$, when n is odd.

Case 1: *n* **odd**

In this case each of the $2n$ vertices of D_2C_n has degree 4 with eccentricity $\frac{n-1}{2}$. So DE matrix of D_2C_n is given by

$$DE(D_2C_n) = \frac{2}{n+7}J_{2n}$$

Since $DE(D_2C_n)$ is a matrix with rank 1, the corresponding DE-spectrum is given by

$$DE\ S_p(D_2C_n) = \begin{pmatrix} \frac{4n}{n+7} & 0 \\ 1 & 2n-1 \end{pmatrix}$$

Case 2: *n* **even**

Here also the graph D_2C_n is 4 regular with eccentricity of each vertex as $\frac{n}{2}$. So the DE matrix is given by

$$DE\ (D_2C_n) = \frac{2}{n+8}\ J_{2n}$$

Hence the DE-spectrum is given by

$$DE\ S_p(D_2C_n) = \begin{pmatrix} \frac{4n}{n+8} & 0 \\ 1 & 2n-1 \end{pmatrix}$$

With all above information, we shall conclude the following proposition.

Proposition 5.4. For the cycle C_n, $n \geq 3$

$$DE\ \varepsilon(D_2C_n) = \begin{cases} \frac{4n}{n+7}, when\ n\ is\ odd \\ \frac{4n}{n+8}\ , when\ n\ is\ even \end{cases}$$

Remark 5.1. For any even integer n ≥ 3,

$$DE\varepsilon(D_2C_{n+1}) - DE\varepsilon(D_2C_n) = \frac{4}{n+8}$$

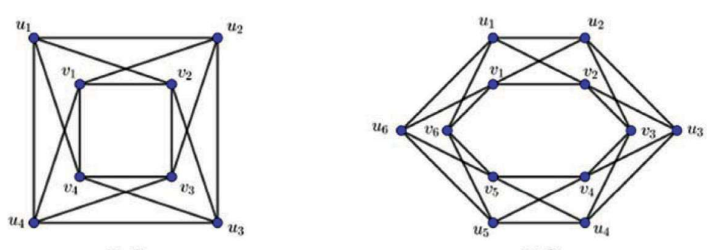

Figure 4: Double graph of C_4 and C_6

We observe that,

$$DE\ (D_2C_4)= \frac{1}{6}J_8 \text{ and } DE(D_2C_6) = \frac{1}{7}J_{12}$$

5.5.2 DE-Spectrum and Energy of Double graph of Star $K_{1,n}$, $n \geq 1$

$K_{1,n}$, $n\geq1$ has n+1 vertices and n edges. Label the central vertex as v and the pendant vertices as $v_1, v_2, \ldots, v_n$. Take another copy of $K_{1,n}$ and label the central vertex as u and the corresponding pendent vertices as $u_1, u_2, \ldots, u_n$. Since v is adjacent to v_i, i= 1, 2, ……, n, in the double graph of $K_{1,n}$, v and u are adjacent to v_i, u_i, $i = 1, 2, \ldots, n$. Then degree of v and u are $2n$ and that of v_i, u_i is 2, $\forall$ i = 1, 2, ….., n. Eccentricity of all the vertices is 2.

The degree sequence of $D_2K_{1,n}$ is (2n, 2n, 2, 2, ….., 2), where 2 is repeating $2n$ times and the corresponding sequence of eccentricities is (2,2, ….., 2). Then the $DE\ (D_2K_{1,n})$ is a rank 1 matrix, which can be written as

$$DE\ (D_2K_{1,n}) = \begin{bmatrix} \frac{1}{2n+2}J_{2\times(2n+2)} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{1}{4}J_{2n\times(2n+2)} \end{bmatrix}$$

The DE-spectrum of $D_2K_{1,n}$ is given by

$$DE\ S_p(D_2K_{1,n})= \begin{pmatrix} \frac{n^2+n+2}{2(n+1)} & 0 \\ 1 & 2n+1 \end{pmatrix}$$

As a result, the argument that follows can be made.

Proposition 5.5. *For n≥1,*

$$DE\ \varepsilon(K_{1,n}) = \frac{n^2+n+2}{2(n+1)}$$

An illustration is displayed below.

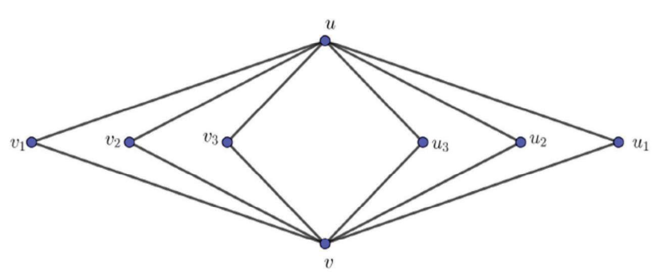

Figure 5: Double graph of $K_{1,3}$

Here $DE\ \varepsilon(K_{1,3}) = \frac{7}{4}$

5.5.3. DE-Spectrum and Energy of Double graph of Friendship graph

The friendship graph F_n is obtained by coalescence n copies of the cycle graph C_3 of length 3 with a common vertex. F_n, $n \geq 2$ has 2n + 1 vertices and 3n edges. [1].

Here we construct the double graph of F_n and DE-spectrum of double graph of F_n. In F_n, there is a vertex with degree $2n$ and all other vertices are of degree 2. Label the vertex with degree $2n$ as v and the remaining vertices as $v_1, v_2, \dots, v_{2n}$. In order to construct the double graph of F_n, take a copy of F_n and label the vertices as u, which was labeled as v in F_n and the remaining as $u_1, u_2, \dots, u_{2n}$. In D_2F_n, u is adjacent to v_i, $i = 1, 2, \dots, 2n$ and v is adjacent to u_i, $i = 1, 2, \dots, 2n$. Hence degree of u, v is $4n$ and that of u_i, v_i is 3, $i = 1, 2, \dots, 2n$. In D_2F_n, eccentricity of each vertex is 2. The degree sequence of D_2F_n is $(4n, 4n, 3, 3, \dots, 3)$ and the corresponding sequence of eccentricity is $(2, 2, \dots, 2)$.

Then by definition of DE matrix, it can be written as

$$\boldsymbol{DE(D_2F_n)} = \begin{bmatrix} \dfrac{\mathbf{1}}{\mathbf{4n+2}} \boldsymbol{J_{2\times(4n+2)}} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{\mathbf{1}}{\mathbf{5}} \boldsymbol{J_{4n\times(4n+2)}} \end{bmatrix}$$

Also the DE-spectrum of F_n is given by,

$$DE\ S_p(D_2F_n) = \begin{pmatrix} \dfrac{8n^2+4n+5}{5(2n+1)} & 0 \\ 1 & 4n+1 \end{pmatrix}$$

So, the following proposition holds.

Proposition 5.6. *For $n \geq 1$,*

$$DE\varepsilon(D_2F_n) = \frac{8n^2+4n+5}{5(2n+1)}$$

An example is illustrated below.

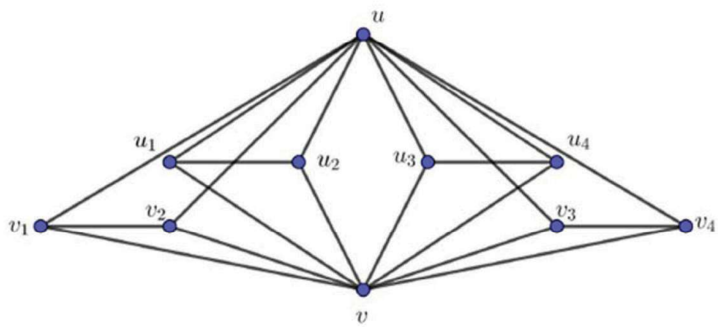

Figure 6: Double graph of F_2

Next we focus on the spectrum of complement of a cycle. For n = 3, 4, we can see that $\overline{\boldsymbol{Cn}}$ results into a disconnected graph. Therefore we confine our discussion to n ≥ 5.

Proposition 5.7. *For n≥5,*

$$DE\ S_p(\overline{C_n}) = \begin{pmatrix} \dfrac{n}{n-1} & 0 \\ 1 & n-1 \end{pmatrix}$$

Proof. Since C_n is a regular graph on n vertices with regularity equal to 2, C_n is *(n−3)*- regular and has eccentricity 2. Therefore each entry of the DE matrix of $\overline{C_n}$ will be $\frac{1}{n-1}$. Clearly DE ($\overline{C_n}$) is of rank 1. Hence DE-spectrum of $\overline{C_n}$ has the eigenvalue 0 with multiplicity *n-1* and trace ($\overline{C_n}$) = $\frac{n}{n-1}$ with multiplicity 1. This completes the proof.

Proposition 5.8. *For the graphs G_1 and G_2, which are not complete, having order m and n respectively*

$$DE\ S_p(G_1 \vee G_2) = \begin{pmatrix} \displaystyle\sum_{i=1}^{m} \frac{1}{d_i + n + 2} + \sum_{j=1}^{n} \frac{1}{d^*{}_j + m + 2} & 0 \\ 1 & m+n-1 \end{pmatrix}$$

Proof. Consider the two graphs G_1 and G_2 with $|G_1| = m$ and $|G_2| = n$. Let d_i, *i = 1, 2, ….., m* and d_j^*, *j*=1,2,…,n be the vertex degrees of G_1 and G_2 respectively. Let $G_1 \vee G_2$ denote the join of two graphs G_1 and G_2. Then $G_1 \vee G_2$ has $m + n$ vertices with degrees $d_i + n$, $d^*{}_j + m$, *i=1,2,….,m, j=1,2,….,n*. Since every vertex of G_1 is adjacent to every vertex of G_2 *and vice versa,* vertices of $G_1 \vee G_2$ has eccentricity as 2. Hence, entries of DE matrix of $G_1 \vee G_2$ are $\frac{1}{d_i+n+2}$ and $\frac{1}{d_j^* + m + 2}$, *i=1, 2, …., m, j = 1, 2, ….., n*. Clearly, DE ($G_1 \vee G_2$) is a rank 1 matrix. This completes the proof.

6. Graphs with DE Energy 1

From our prior explanation we can see that the complete graph is one of the examples of standard graph with DE energy equal to 1. In this section we discuss graphs with DE energy 1.

6.1. Construction of $C_{2,n}$

Here we discuss a procedure for constructing an $(n-2)$ -regular graph on n vertices with eccentricity of each vertex as 2, $C_{2,n}$, where n is a positive even integer, $n \geq 4$. Denote this graph by $C_{2,n}$. The construction of $C_{2,n}$ begins with an n-cycle C_n, whose vertices are consecutively labeled $v_1, v_2, \ldots, v_n$ clockwise around its perimeter as in the figure given below.

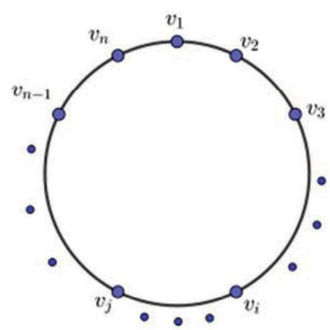

Figure 7: Construction of $C_{2,n}$ starting with C_n

For two vertices v_i and v_j, $i \neq j$, the adjacency in the graph $C_{2,n}$ is determined by the distance between v_i and v_j along the perimeter of the cycle C_n. Since n is a positive even integer, there exists a positive integer k with $n = 2k$. Two vertices v_i and v_j are adjacent, if their distance is not equal to k.

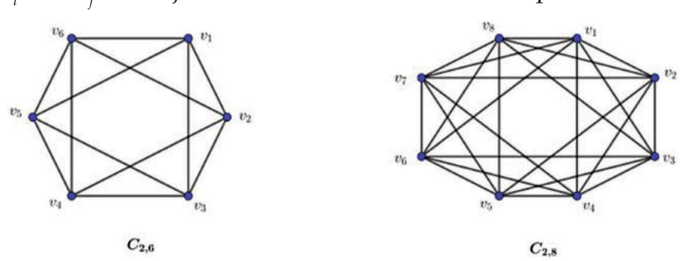

Figure 8: $C_{2,6}$ and $C_{2,8}$

We can observe from the construction itself that $C_{2,n}$ is an $n-2$ regular graph with eccentricity of each vertex as 2.

Also $DE\ (C_{2,n}) = \frac{1}{n} J_n$, where J_n is n x n matrix with all of its entries as 1. So $C_{2,n}$ has the same DE spectrum that of complete graph K_n. Hence by Proposition 5.2, DE ε $(C_{2,n}) = 1$.

Next, we write an algorithm for the above said, construction.

Algorithm

Construction of an $(n-2)$ -regular graph of order n and eccentricity 2

Input : Positive integers n and k with $n = 2k$.
Output : The graph $C_{2,n}$ with vertex labels $v_1, v_2, \ldots, v_n$.
Initialize graph $C_{2,n}$ to be n isolated vertices with labels $v_1, v_2, \ldots, v_n$.
Let $n = 2k$.
For $i = 1$ to $\frac{n}{2}$
If $j = i + k$
Create an edge between all other vertices except v_i and v_j.
Return graph $C_{2,n}$.
For $i = \frac{n}{2} + 1$ to n
If $j = i + k - n$
Create an edge between all other vertices except v_i and v_j.
Return graph $C_{2,n}$.

Proposition 6.1. *For n≥4, DE* $\varepsilon(c_{2,n}) = 1$.

Proposition 6.2. For n≥4, DE ε $(C_{2,n} \vee C_{2,n}) = 1$

Proof. $C_{2,n} \vee C_{2,n}$ is a graph with order $2n$ and $2n$-2 regular. Eccentricity of each vertex in $C_{2,n} \vee C_{2,n}$ is 2. So the degree sequence and the corresponding eccentricity sequence is $(2n - 2, 2n - 2, \ldots, 2n - 2)$ and $(2,2,\ldots,2)$ respectively. Therefore,

$$DE\left(C_{2,n} \vee C_{2,n}\right) = \frac{1}{2n} J_{2n}$$

Clearly the DE matrix of $C_{2,n} \vee C_{2,n}$ is of rank 1, the corresponding DE-spectrum is given by,

$$DE\ S_p\left(C_{2,n} \vee C_{2,n}\right) = \begin{pmatrix} 1 & 0 \\ 1 & 2n-1 \end{pmatrix}$$

Hence $DE\ \varepsilon\ \left(C_{2,n} \vee C_{2,n}\right) = 1$.

7. An Upperbound for the Eigenvalues of DE Matrix

This section provides us with the upperbound for the eigenvalues of DE matrix in terms of the spectral radius of the given graph G, which is a more better upperbound than the spectral radius of the corresponding matrix.

Theorem 7.1. ***[13]*** *(Hopf). For a positive matrix of* $A = (a_{ij})$ *order n,*

$$\alpha = \max\{a_{ij} \mid 1 \le i, j \le n\}, \qquad \beta = \min\{a_{ij} \mid 1 \le i, j \le n\}.$$

If λ *is an eigenvalue of the matrix A other than* $\rho(A)$*, then*

$$|\lambda| \le \frac{\alpha - \beta}{\alpha + \beta}\rho(A)$$

Proposition 7.1. *Let the order of the graph G be n and* $n \ge 2$*. If* λ *is an eigenvalue of DE(G) other than* $\rho(DE(G))$*, then*

$$|\lambda| \le \left(1 - \frac{2}{n}\right)\rho(DE(G))$$

Proof: Proof of this proposition holds directly from Theorem 7.1 and with the fact that for *DE(G),* $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{2n-2}$.

Consider the graph P_4.

Here $\lambda_1 = \rho(DE(P_4)) = 1.0164$ and $\lambda_2 = 0.0164$, $\lambda_3 = \lambda_4 = 0$.

By Proposition 7.1, for $i = 2, 3, 4$

$$|\lambda_i| = \left(1 - \frac{2}{4}\right) \times 1.0164 \le 0.5082$$

8. Concluding Remarks

The DE matrix is a new kind of positive matrix derived from a given graph. We have shown that DE matrix is irreducible and primitive. Also determined the DE-spectrum and energy of different classes of graphs and graph operations. Further, this matrix can be extended to other areas of mathematics and fields of Sciences. In future one can find so many applications related to this matrix.

Conflicts of Interest. Regarding the publishing of this work, the authors state that they have no conflicts of interest.

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Declarations

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