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An accelerating flat universe with an inverse square variation of the equation of state parameter with scale factor

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Abstract

The pressure parameter in the equation of state (EoS) of the cosmic fluid that varies inversely as the square of the scale factor is a possibility to explain the evolution of our flat universe, presently expanding with an increasing acceleration under the negative pressure of a dynamic dark energy. The proposed model demonstrates the switch over from a state of decelerating expansion to a state of accelerating expansion of the universe using a simple equation of state consistent with the cosmological observations by introducing an effective pressure parameter in the equation of state for the cosmic fluid which comprises of radiation, matter and dark energy.

Keywords: Accelerating expansion, Hubble parameter, Pressure parameter, Dark energy.

MSC 2020: 85A40.

1. Introduction

The standard model in cosmology is also called big-bang model and the theoretical framework is formulated based on cosmological principle of homogeneity and isotropy. Based on this assumption Alexander Friedmann solved Einstein's field equations using the spatially symmetric Robertson-Walker metric and proved that universe expands uniformly but at a decelerating rate [1, 2]. However, in 1998, observations associated with Type Ia supernovae suggested that this expansion is speeding up rather than decelerating [3, 4]. Thereafter, several observations supported the idea of accelerating cosmic expansion [5–8].

The cause of this acceleration is unknown and is a great problem faced by cosmologists. The Friedmann equations as solutions to Einstein's

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field equations describe the dynamics of the universe in terms of a few cosmological parameters like scale factor, density and pressure of the fluid as the source term for gravity. This fluid called cosmic fluid is considered as a perfect non-viscous fluid having many components like matter, radiation etc. The equation of state $p=\omega\rho c^2$ relates pressure to the density of the fluid component. Here, ω is called pressure parameter or omega parameter. One can show that, if we include a component with negative pressure in the cosmic fluid, then expansion rate can speed up if $\omega < -\frac{1}{3}$. Hence, it is believed that there is a hidden form of mysterious energy with negative pressure called dark energy exists in the universe, which account for the recent cosmic acceleration observed. Many recent observations support that, approximately 70% of the energy density of the universe is in this form with negative pressure to make theory compatible with recent observations [9-12]. A dark energy with equation of state $p = -\rho c^2$, with pressure parameter $\omega = -1$ is known as the cosmological constant (Λ), since the corresponding energy density remains constant even when volume increases. Such a constant dark energy can produce accelerated expansion of the universe. However, such models (commonly called ΛCDM or Lambda Cold Dark Matter Model) has many theoretical difficulties. The fine-tuning issue arises from the observation that the measured value of Λ is about 120 orders of magnitude less than the value anticipated by quantum field theory for vacuum energy density, requiring theoretical modifications that are not physically justified [13, 14]. The coincidence problem in cosmology - a coincidence that lacks a natural explanation within the Λ Cold Dark Matter (ΛCDM) model [15]. Additionally, there is the synchronicity problem, which asks why cosmic acceleration commenced recently in cosmic history, rather than at an earlier epoch. These conceptual difficulties drive the exploration of alternative or dynamic models of dark energy, including those with a time-varying equation of state parameter ω, scalar field models, or modified gravity theories [16–19].

Also, the growing tensions in Hubble parameter measurements [20–22] and large-scale structure observations have further motivated the exploration of dynamical dark energy models [23–26].

In this work, we introduce a phenomenological model for the total cosmic fluid that incorporates radiation, matter, and dark energy through an effective equation of state parameter ω_{eff}, which evolves dynamically with the scale factor R(t). This model aims to provide a smooth transition from early decelerated expansion (radiation-and matter-dominated) to the current accelerated phase, consistent with constraints from recent observational data.

The theoretical foundation is built upon the Einstein field equations [27]:

$$R_{ik} - \frac{1}{2} g_{ik} R = -\frac{8\pi G}{c^4} T_{ik} \tag{1}$$

Where, R_{ik} is the Ricci tensor, g_{ik} is the metric tensor, R is the Ricci scalar and T_{ik} is the energy momentum tensor of the cosmic fluid, given by $T_{ik} = (P+\rho c^2) u_i u_k - P g_{ik}$. u_i is the four-velocity vector. Friedmann assumed the validity of the cosmological principle to derive the Friedmann equations. So, he used the Robertson-Walker metric and arrived at two equations called Friedmann equations [2, 27, 28]

$$\frac{\dot{R}^2}{R^2} + \frac{kc^2}{R^2} = \frac{8\pi G \rho(t)}{3} \tag{2}$$

$$2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{kc^2}{R^2} = -\frac{8\pi G P(t)}{c^2} \tag{3}$$

Here R(t) is the scale factor, which completely describes the evolution dynamics of the universe. ρ(t) and P(t) are the energy density and pressure of the cosmic fluid, and k is the curvature parameter characterizing the spatial geometry of the four-space. Here the first equation represents the velocity and the second one describes the acceleration of the expansion. The equation of state parameter is defined as [2]:

$$\omega = \frac{P}{\rho c^2} \tag{4}$$

Here $\omega = \frac{1}{3}$ for radiation, $\omega = 0$ for matter and $\omega = -1$ for dark energy in the form of cosmological constant. If the cosmic fluid representing the universe has $\omega > -\frac{1}{3}$, then universe will expand with deceleration or with acceleration when $\omega < -\frac{1}{3}$. If, $\omega < -\frac{1}{3}$, then universe expands uniformly and is called coasting evolution. This evolution dynamics follows from the combined Friedmann equation, $\ddot{R} = -\frac{4\pi G}{3} R(1 + 3\omega)\rho$ [2]. If we take the assumption that, each component in the cosmic fluid is separately conserved, which is reasonable since during different eras, universe is dominated by a single component, then by energy conservation law, $dU+PdV = 0$. So, from Friedmann equations, we get, density $\rho \propto R^{-3(1+\omega)}$ [28]. $\rho \propto R^{-4}$ for radiation ($\omega = 1/3$), $\rho \propto R^{-3}$ for pressure-less matter ($\omega = 0$), $\rho = constant$ for dark energy with $\omega = -1$.

In this study, we assume a scale-factor-dependent effective equation of state parameter ω, which is probably due to a dynamic dark energy and hence the evolution naturally connects between various epochs dominated by different components in the fluid. A large number of dynamic dark energy models are proposed in the literature to explain the recent cosmic observations and has greater flexibility in analysing the cosmological data compared with modified field theoretic approaches [29–31].

The present model is motivated by some recent works on, parametrized dark energy or effective equation of state with an evolving or dynamic omega parameter [32–37], possible scalar field theories [38–40], interacting and viscous fluids [41], and modified gravity models of curvature and torsion [42–45]. In particular, recent works such as those by Odintsov et al. [46] and Abdalla et al. [47] have emphasized the importance of designing EoS models that remain compatible with Planck 2018 data and BAO results while consistent with late-time dynamics of the cosmos. The results of Supernova Cosmology Project (SCP) established the fact that the expansion rate of the universe is speeding up and is happening at least 5 billion years ago. Also, the acceleration of the universe as observed, needs the negative pressure parameter for dark energy to be at least 0.6 in magnitude [48]. Using the apparent magnitude-redshift data for the distant Type Ia supernova, the Hubble parameter has been tuned [49] to $2.173\times10^{-18}/s$.

In Section 2, we present the mathematical formulation of our proposed model with a scale-factor-dependent ω. Section 3 discusses the resulting cosmic dynamics and their compatibility with current observations. Section 4 summarizes our conclusions and outlines possible extensions, such as the inclusion of perturbation analysis or comparison with future data from DESI and Euclid missions.

2. New Model and the Method

The universe is approximated as a non-viscous cosmic fluid consists of radiation, matter and dark energy and ρ has all these three components in it. If total energy is conserved, then, we need to use an effective equation of state parameter ω_{eff} . A fluctuating equation-of-state parameter, leading to stochastic evolution of early universe was also proposed in the literature [50]. In the present work, we consider a rather late evolution of the universe and hence cosmic fluid is treated as a two-component fluid - a fluid of only matter and dark energy (as the effect of radiation can be safely turned off). So, we propose a simple effective equation of state parameter for the whole fluid which varies inversely as R^2. Let,

$$\omega_{eff} = -\frac{1}{3}\left(1+\frac{F}{R^2}\right), F = R^2 - R_0^2 \tag{5}$$

Since galaxies are moving away from each other in an expanding universe, the wavelength of light from each galaxy is measured to be shifted towards the red end of the electromagnetic spectrum. The measurement of redshift is direct evidence for the expansion of the universe. Let scale factor R(t) corresponds to an epoch of redshift z. R_0 is the scale factor of the universe at the transition redshift. If z_0 is the redshift corresponding to R_0, then it can be related to the present scale factor by [2],

$$1 + z = \frac{R_p}{R} \tag{6}$$

Where R_p is the present scale factor, and

$$1 + z_0 = \frac{R_p}{R_0} \tag{7}$$

So, effective pressure parameter is

$$\omega_{eff}(z) = -\frac{2}{3} + \frac{1}{3}\left[\frac{1+z}{1+z_0}\right]^2 \tag{8}$$

This represents the dynamic evolution of the effective equation of state of the cosmic fluid comprising of matter and the mysterious dark energy. At the transition epoch z=z_0, we get, $\omega_{eff}(z_0) = -\frac{1}{3}$, which marks the boundary between deceleration and acceleration, as expected. For $z < z_0$, $\omega_{eff} > -\frac{1}{3}$ ⇒ decelerating universe. For $z < z_0$, $\omega_{eff} < -\frac{1}{3}$ ⇒ accelerating universe. This is illustrated in Fig. 1, where effective equation of state is plotted as a function of redshift.

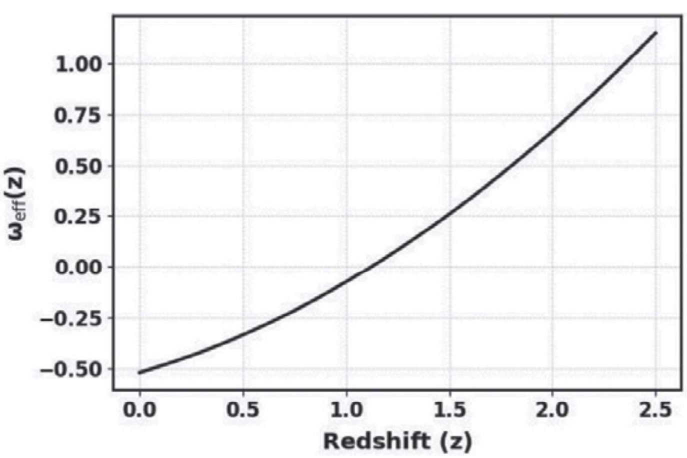

Figure 1. Evolution of the effective equation-of-state parameter ω_{eff} as a function of redshift. The parameter transitions from negative values at low z (accelerated expansion) to positive values at higher z (decelerated expansion).

If we subtract Eq. (2) from Eq. (3) and use Eq. (5), we get,

$$2\frac{\ddot{R}}{R}+\frac{8\pi G\rho}{3}-\frac{8\pi G\rho}{3}\left(2-\frac{R_0^2}{R^2}\right)=0 \tag{9}$$

To get an order of estimate of relevant cosmological parameters, we start from a known value of present Hubble parameter $H_p = 2.173\times10^{-18}/s$. Now, simplifying Eq. (9), we get acceleration equation in our model as,

$$\ddot{R}=\frac{4\pi GF\rho}{3R} \tag{10}$$

So, $\ddot{R}<0, F<0$ and $\ddot{R}>0, F>0$.

By energy conservation equation [2], we get

$$d(R^3\rho)-\frac{1}{3}\left[2-\left(\frac{R_0}{R}\right)^2\right]\rho d(R^3)=0 \tag{11}$$

$$\frac{d\rho}{\rho}+\frac{dR}{R}+R_0^2\frac{dR}{R^3}=0 \tag{12}$$

Integrating we get

$$log_e\left(\frac{\rho}{\rho_i}\right)=log_e\left(\frac{R_i}{R}\right)+log_e\left(\exp\left(\frac{R_0^2}{2}\left[\frac{1}{R^2}-\frac{1}{R_i^2}\right]\right)\right) \tag{13}$$

R_i represents the expansion factor of the universe at the beginning of matter era, when matter started to dominate over radiation (which is nearly ten thousand years according to standard cosmological model) compared to the age of the universe and ρ_i the initial density of the cosmic fluid which is a solution of matter and dark energy.

$$P=-\frac{c^2}{3}\left(2-\frac{R_0^2}{R^2}\right)\frac{\rho_i R_i}{R}\exp\left(\frac{R_0^2}{2}\left[\frac{1}{R^2}-\frac{1}{R_i^2}\right]\right) \tag{18}$$

3. Results from the New Model

The dynamics and state of the universe is described through the values of cosmological parameters at various epochs. In the present model, the cosmological parameters are evaluated with $R = R_i$ in Eq. (5) with $\omega_{eff} = 0$ gives the scale factor at transition as,

$$R_0 = 1.414\ R_i \tag{19}$$

When $R = R_0, \omega_{eff} = -\frac{1}{3}$ and as $R \to \infty, \omega_{eff} \to -\frac{2}{3}$. Table 1 represents values of the effective equation of state at different epochs characterized by different scale factors.

Table 1: Effective pressure parameter and corresponding scale factor

ω_{eff}	R(t)
0	R_i
$-\frac{1}{3}$	R_0
$-\frac{2}{3}$	∞

It is to be noted that at $R = R_0, \omega_{eff} = -\frac{1}{3}$ which justifies our postulate of the equation of state parameter. If the present value of ω_{eff} is taken as -0.5,

$$\frac{R_p}{R_0} = 1.414 \quad (20)$$

Then, present scale factor can be expressed as,

$$R_p = 2.0R_i \quad (21)$$

Hubble parameter becomes,

$$H_p = \frac{c}{\sqrt{R_p R_i}} \exp\left(\frac{R_0^2}{4}\left[\frac{1}{R_p^2} - \frac{1}{R_i^2}\right]\right) \quad (22)$$

Substituting present H_p,

$R_p = 1.342 \times 10^{26}$

(Comparable with the 10^{26} of the standard model).

Initial state of the universe (After radiation era):

$R_i = 0.67 \times 10^{26}$

$$H_i = \frac{c}{R_i} = 4.4776 \times 10^{-18}/s$$

$$\rho_i = \frac{3H_i^2}{8\pi G} = 3.589 \times 10^{-26} kgm^{-3}$$

$P_i = 0$

$\omega_{eff} = 0$

State of the universe at the transition shift:

$R_0 = 0.949 \times 10^{26}$
$H_0 = 2.928 \times 10^{-18}/s$
$\rho_0 = 1.535 \times 10^{-26}\ kgm^{-3}$
$P_0 = -0.333\ \rho_0 c^2 = -4.60 \times 10^{-10}\ Pa$
$\omega_0 = -0.333$

Present state of the universe:

$R_p = 1.342 \times 10^{26}$
$H_p = 2.173 \times 10^{-18}/s$
$\rho_p = 0.8455 \times 10^{-26}\ kgm^{-3}$
$P_p = -3.805 \times 10^{-10}\ Pa$
$\omega_{eff} = -0.500$

Final state of the universe:

$R_f = \infty$
$H_f = 0$
$\rho_f = 0$
$P_f = -0$
$\omega_{eff} = -0.666$

The model allows all cosmological parameters- scale factor, Hubble parameter, mass density and pressure to be expressed as a function of ω_{eff} given by,

$$R(\omega_{eff}) = \frac{R_0}{\sqrt{2+3\omega_{eff}}} \tag{23}$$

$$H(\omega_{eff}) = \frac{c}{\sqrt{R_0 R_i}}(2+3\omega_{eff})^{\frac{1}{4}} \exp(0.25[2+3\omega_{eff}] - 0.5) \tag{24}$$

$$\rho(\omega_{eff}) = \frac{3c^2}{8\pi G R_0 R_i}(2+3\omega_{eff})^{\frac{1}{2}} \exp(0.5[2+3\omega_{eff}] - 1.0) \tag{25}$$

$$P(\omega_{eff}) = \frac{3\omega_{eff} c^4}{8\pi G R_0 R_i}(2+3\omega_{eff})^{\frac{1}{2}} \exp(0.5[2+3\omega_{eff}] - 1.0) \tag{26}$$

Integral of Hubble's law:

$$\int \sqrt{\frac{1}{R}} \exp\left(-\frac{R_0^2}{4R^2}\right) dR = 2.247 \times 10^{-5} \int dt \tag{27}$$

approximating it for $R \to \infty$, $\sqrt{R_2} \approx \sqrt{R_1} + 1.123 \times 10^{-5}(t_2 - t_1)$.

4. Generalization

The effective equation of state parameter is the ratio of the total pressure to total energy density of the cosmic fluid comprising of matter, dark matter, radiation and dark energy and is proposed to evolve with scale factor or redshift. A dynamic EoS such as given by Eq. (8) smoothly transitions from a matter-dominated phase (ω=0) at high redshift to an accelerated expansion (ω<−1/3) at low redshift, with z_0 marking the transition redshift. The Friedmann equations govern the expansion of the universe through the Hubble parameter H(z), which relates to the energy density ρ(z) of the cosmic fluid for a flat universe via [2]:

$$H^2(z) = \frac{8\pi G}{3}\rho(z) \tag{28}$$

The energy conservation equation is [2],

$$\dot{\rho} + 3H(\rho + \frac{P}{c^2}) \tag{29}$$

With a dynamic EoS parameter $\omega_{eff} = \frac{P}{\rho c^2}$,

$$\dot{\rho} = -3H\rho(1 + \omega_{eff}) \tag{30}$$

Using Eq. (6)

$$\frac{dz}{dt} = -(1 + z)H \tag{31}$$

So,

$$\frac{dt}{dz} = -\frac{1}{(1+z)H} \tag{32}$$

Since, $\frac{d\rho}{dz} = \frac{d\rho}{dt}\frac{dt}{dz}$ and substitute from Eq. (30), we get

$$\frac{d\rho}{dz} = \frac{3\left(1+\omega_{eff}(z)\right)}{(1+z)}\rho(z) \tag{33}$$

From this equation, one can determine ρ(z), and hence H(z). This in turn allows the computation of the deceleration parameter. The deceleration parameter q(z) is a fundamental cosmological parameter that measures whether the expansion of the universe is speeding up or slowing down. Defined as,

$q = -\frac{\ddot{R}R}{\dot{R}^2}$ [2, 28], it indicates whether the expansion is accelerating (q < 0) or decelerating (q > 0). If the universe is dominated by matter (non-relativistic as well as relativistic), cosmic fluid has positive pressure and hence universe undergoes decelerated expansion (q > 0). If the universe is dominated by dark energy with negative pressure, it undergoes an accelerated expansion (q > 0). Thus, the evolution of the deceleration parameter as a function of redshift describes the evolution history and dynamics of the universe. The transition from deceleration to acceleration is characterized by a zero value of the deceleration parameter. One can express q in terms of *H*. From the formula of deceleration parameter and Hubble parameter given above, one can express q as,

$$q = -\frac{\ddot{R}}{RH^2} \tag{34}$$

Also,

$$\dot{H} = \frac{\ddot{R}}{R} - \frac{\dot{R}^2}{R^2} \tag{35}$$

Substitute for $\frac{\ddot{R}}{R}$ from Eq. (35) into Eq. (34),

$$q = -1 - \frac{\dot{H}}{H^2} \tag{36}$$

In terms of redshift, using Eq. (31), we get,

$$\dot{H} = \frac{dH}{dz}\frac{dz}{dt} = -(1+z)H\frac{dH}{dz} \tag{37}$$

So,

$$\frac{\dot{H}}{H^2} = -(1+z)\frac{1}{H}\frac{dH}{dz} = -(1+z)\frac{dlnH}{dz} \tag{38}$$

Using, $\frac{dlnH}{dln(1+z)} = (1+z)\frac{dlnH}{dz}$, Eq. (36) becomes

$$q(z) = -1 + \frac{dlnH(z)}{dln(1+z)} \tag{39}$$

In a flat universe ($H^2 = \frac{8\pi G}{3}\rho$), the deceleration parameter in terms of the effective EoS is derived from the Raychaudhuri equation [2]: $\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\rho(1 + 3\omega_{eff})$,

$$q(z) = -\frac{\ddot{R}}{RH^2} = \frac{1}{2}(1 + 3\omega_{eff}(z)) \tag{40}$$

Together, H(z) and q(z) provide insights into the universe's expansion history and acceleration behaviour. Using Eqs. (28) and (33), we get for the present model,

$$H(z) = H_0\sqrt{(1+z)\exp(\frac{(1+z)^2}{2(1+z_0)^2})} \tag{41}$$

And,

$$q(z) = \frac{1}{2}([\frac{1+z}{1+z_0}]^2 - 1) \tag{42}$$

The evolution of Hubble parameter and deceleration parameter as a function of redshift is illustrated in Fig. 2 and Fig. 3 respectively.

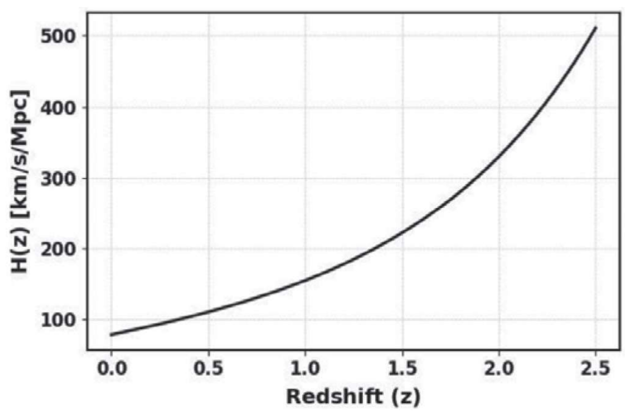

Figure 2. Redshift evolution of the Hubble parameter *H* (*z*). The monotonic increase reflects the higher expansion rate of the Universe at earlier epochs.

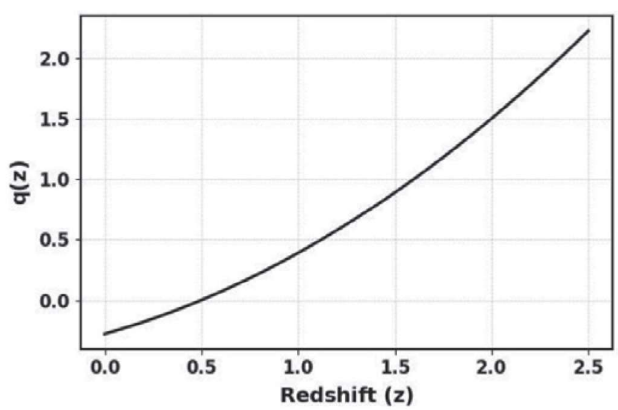

Figure 3. Evolution of the deceleration parameter *q*(z). The transition from positive to negative values indicates a shift from cosmic deceleration at high redshift to acceleration at lower redshift. The transition occurs around z~0.5 as supported by recent observations.

5. Conclusions and future scope

Our model is a bigger-two component fluid whose size changes as a parabola, with the simple equation of state Eq. (5) that was, at the end of the radiation era in a state of zero pressure and size 0.67×10^{26} with 3.589×10^{-26} Kg of mass per unit volume and was expanding with a speed of 4.4776×10^{-18} /s; dark energy which must be thought to be closely connected to the density of matter and also the intrinsic topological properties of space time gradually made tensor forces in the fabric of space-time strong enough to suddenly make the deceleration change to acceleration at the effective pressure parameter $-\frac{1}{3}$ when universe was 0.949×10^{26} in size. Presently the mass density of the universe is 0.8455×10^{-26}, speed 2.173×10^{-18} /s and pressure 3.805×10^{-10} Pa. Dark energy density along with the density of matter is gradually coming down to zero but is going to be in the universe till the end of time when the pressure parameter tends to $-\frac{2}{3}$.

The new equation of state is a simple one that cleverly demonstrates the phase-shift of the universe from deceleration to acceleration within the frame of the current understanding about our universe and the concept of effective pressure parameter successfully describes the evolution dynamics of the universe. The dynamic effective equation of state parameter proposed here, for the mixture of matter and dark energy that has not been addressed seriously in literature can be a powerful alternative to the pressure parameter of the standard model to represent the true nature of the late-time evolution of the cosmos.

The present model with an evolving effective pressure parameter can successfully explain the late-time cosmic evolution. It favours a dynamic dark energy parametrization. It is a simple, phenomenological model aimed at explaining the switch over from decelerated to accelerated expansion of the universe which occurred at about five billion years ago. However, a more detailed analysis of the model with observational data is essential to constrain the model parameters effectively. Second, the physical origin and theoretical interpretation of the evolving dark energy component deserve further investigation-possibly through scalar field models or modified gravity models through the concepts of torsion or modified Ricci scalar for curvature or combined. This can provide a better understanding of the dynamics of the universe and the nature of the source of energy responsible for the observed late cosmic acceleration.

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Author Contributions

Both authors contributed equally.

References

[1]. P. J. E. Peebles: *Principles of Physical Cosmology*, Princeton University Press, Princeton, 1993.

[2]. J. V. Narlikar: *Introduction to cosmology*, Cambridge university press, Cambridge, 1993.

[3]. A. G. Riess et al. *Observational evidence from supernovae for an accelerating universe and a cosmological constant*, Astronomical Journal 116: 1009-1038, 1998.

[4]. S. Perlmutter et al. Ap. J. 517: 565-586, 1999.

[5]. N. Aghanim et al. *Planck 2018 results: Cosmological parameters*, Astron. Astrophys. 641: A6, 2020.

[6]. S. Alam et al. Phys. Rev. D 103: 083533, 2021.

[7]. E. Tomasetti et al. *Constraints on dynamical dark energy models using the latest data*, Astron. Astrophys. 679: A96, 2023.

[8]. R. Jimenez et al. *Updated observational constraints on H(z) measurements and the Hubble tension*, JCAP 11: 047, 2023.

[9]. M. Li et al. *Dark energy*, Commun. Theor. Phys. 56: 525-604, 2011.

[10]. G. Gu et al. *Observational signatures of dynamical DE models*, RAA 24: 065001, 2024.

[11]. T. Padmanabhan: *Cosmological constant*, Phys. Rep. 380: 235-320, 2003.

[12]. W. Giare et al. *A cosmological analysis of dark energy evolution*, JCAP 10: 035, 2024.

[13]. S. Weinberg: *The cosmological constant problem*, Rev. Mod. Phys. 61: 1-23, 1989.

[14]. J. Martin: *Everything you always wanted to know about the cosmological constant problem*, Comp. Ren. Phys. 13: 566-665, 2012.

[15]. I. Zlatev, L. Wang and P. J. Steinhardt: *Quintessence, cosmic coincidence and the cosmological constant*, Phys. Rev. Lett. 82: 896-899, 1999.

[16]. E. J. Copeland, M. Sami and S. Tsujikawa: *Dynamics of dark energy*, Int. J. Mod. Phys. D 15: 1753-1936, 2006.

[17]. L. K. Duchaniya et al. *Generalized holographic Ricci dark energy models*, Phys. Dark Univ. 44: 101461, 2024.

[18]. S. Nojiri and S. D. Odintsov: Phys. Rep. 505: 59-144, 2011.

[19]. E. D. Valentino et al. Class. Quant. Grav. 38: 153001, 2021.

[20]. A. G. Riess et al. *A comprehensive measurement of the local value of the Hubble constant*. ApJ. Lett. 934: L7, 2022.

[21]. L. R. Colaco et al. *A joint analysis of strong lensing and type Ia supernovae to determine Hubble constant*, Eur. Phys. J. C 85: 577 (2025).

[22]. L. Verde, T. Treu and A. G. Riess: Nat. Astron. 3: 891-895, 2019.

[23]. S. Pan and S. Chakraborty: *A cosmographic analysis of various parametrizations of dark energy*, Int. J. Mod. Phys. D 28: 1950151, 2019.

[24]. W. Wang et al. *Interacting dark energy models after Planck 2018*, JCAP 01: 047, 2024.

[25]. B. J. Barros et al. *Observational constraints on interacting models*, Eur. Phys. J. C 84: 276, 2024.

[26]. S. Capozziello et al. *Modified gravity approaches and cosmological observations*, Symmetry 15: 65, 2023.

[27]. S. Weinberg: *Gravitation and Cosmology: Principles and applications of GTR*, Wiley, New York, 1972.

[28]. V. Mukhanov: *Physical foundations of cosmology*, Cambridge university press, Cambridge, 2005.

[29]. B. Wang et al. *Reconstructing quintessence and tachyon potentials from recent data*, Phys. Rev. D 108: 043523, 2023.

[30]. S. D. Odintsov and V. K. Oikonomou: *Singular cosmology from generalized entropy formalism*, Phys. Rev. D 107: 083519, 2023.

[31]. O. Akarsu et al. *Probing dynamical DE with cosmographic methods*, Phys. Rev. D 107: 123503, 2023.

[32]. Y. Tada and T. Terada: *Primordial black holes from the inflating curvaton*, Phys. Rev. D 109: L121305, 2024.

[33]. M. Cortes and L. R. Liddle: *Bayesian model comparison of cosmologies with dynamical DE*, JCAP 12: 007, 2024.

[34]. L. A. Escamilla et al. *Holographic dark energy in f (R, T) gravity*, JCAP 2024: 091, 2024.

[35]. A. C. Alfano et al. *Late time cosmological dynamics of generalized holographic models*, Phys. Dark Univ. 42: 101298, 2023.

[36]. H. Wang and Y. S. Piao: *Cosmological constant problem in inflationary models*, Phys. Rev. D 105: 083504, 2022.

[37]. F. B. M. dos Santos: *Observational constraints on interacting quintessence models*, JCAP 07: 064, 2023.

[38]. K. Bamba et al. *Dark energy cosmology*, Ap. Space Sci. 342: 155-228, 2012.

[39]. J. Niu et al. *Cosmic growth index in modified gravity*, ApJ. 972: 14, 2024.

[40]. K. V. Berghaus et al. *Planck constraints on extra relativistic species and DE models*, Phys. Rev. D 110: 103524, 2024.

[41]. M. Lucca et al. *Dark sector interactions and dark energy*, Phys. Rev. D 108: 063501, 2023.

[42]. T. Clifton et al. *Modified gravity and cosmology*, Phys. Rep. 513: 1-189, 2012.

[43]. Y. F. Cai et al. *f (T) teleparallel gravity and cosmology*, Rep. Prog. Phys. 79: 106901, 2016.

[44]. M. Krssak et al. *Teleparallel theories of gravity*, Class. Quant Grav. 36: 183001, 2019.

[45]. S. Bahamonde et al. *Teleparallel gravity: From theory to cosmology*, Phys. Rep. 775: 1-122, 2018.

[46]. S. D. Odintsov et al. *A novel exponential f (R) gravity model for dark energy*, Eur. Phys. J. C 85: 298, 2025.

[47]. M. C. B. Abdalla et al., *Dark energy and dark matter interactions: theoretical models and observational constraints*, Phys. Dark Univ. 38: 101256, 2022.

[48]. M. Tristram et al. *Cosmological parameters derived from final Planck data release*, A&A, 682: A37, 2024.

[49]. P. M. Garnavich et al. *Supernova limits on the cosmic equation of state*, ApJ. 509: 74-79, 1998.

[50]. C. Sivakumar, M. V. John and K. Babu Joseph: *Stochastic evolution of cosmological parameters in the early universe*, Pramana 56: 466-477, 2001.