Mapana – Journal of Sciences
2026, Vol. 25, No. 1, *73-82*
ISSN 0975-3303|https://doi.org/10.12723/mjs.76.5

Some Results on Square Prime Cordial Graphs

I. Blessy*and I. Gnanaselvi*

Abstract

Let G = (p, q) be a graph with p vertices and q edges. A SPC labeling of a graph G with vertex collection V(G) is a bijection $\alpha : V(G) \rightarrow \{0, 1, 2, \ldots, p-1\}$ to the extent that its induced binary edge labeling function $\alpha^* : E(G) \rightarrow \{0, 1\}$ is defined by $\alpha^*(uv) = \begin{cases} 0 & \text{if } [((\alpha(u))^2, (\alpha(v))^2) + 2d(u,v)] \equiv 0 \pmod 2 \\ 1 & \text{if } [((\alpha(u))^2, (\alpha(v))^2) + 2d(u,v)] \equiv 1 \pmod 2 \end{cases}$. The difference between the edges designated as 0, denoted by $e_\alpha(0)$ and the edges designated as 1, denoted by $e_\alpha(1)$ is atmost 1. A graph which follows SPC labeling is referred to as SPC graph. This paper elucidates that broom graph $B_{n,m}$ and $K_{1,m}*K_{1,n}$ admits SPC Labeling and some Results.

Keywords: SPC labeling, SPC graph, parity labeling.

1. Introduction:

Graph labeling is one of the most famous topics in graph theory which is nothing but assigning integers to the graph's edges or vertices or both with certain conditions. A dynamic survey on various graph labeling techniques can be found in Gallian. J. A. [1]. We followed the basic symbols and terminologies of graph theory as in F. Harary [2]. The concept of cordial labeling was coined by Cahit [11]. The definition of prime cordial labeling was published by Sundaram et al [10], which investigates the coprimality of the vertex labels. Also, many authors have extended their research in prime cordial labeling for several graphs. We introduce a new variation called Square Prime Cordial labeling, which investigates the parity of the sum of squared GCD and the distance d(u, v) where d(u, v) represents the distance between u and v. Taking advantage of the fact that in this case d(u, v) is always 1. This definition connects the number theoretic properties of integers with graph Theory. The Greatest Common Divisor (GCD) of any two integers x and y can be denoted by (x, y) instead of GCD (x, y) [13].

* Department of Mathematics and Research Centre, Sarah Tucker College, Affiliated to Manonmaniam Sundaranar University, Tirunelveli, Tamil Nadu, India; blessy3231@gmail.com; selvikim@gmail.com

2. Preliminaries

Let G = (p, q) be a graph. A prime cordial labeling of a graph G with vertex collection V(G) is a bijection f : V(G) → {1, 2, …, p} to the extent that its induced binary edge labeling function f * : E(G) → {0, 1} is defined by f *(uv) = $\begin{cases} 0 & \text{if } (f(u), f(v)) > 1 \\ 1 & \text{if } (f(u), f(v)) = 1 \end{cases}$. The number of edges designated as 0, is denoted by e_f (0) and the number of edges designated as 1, is denoted by e_f (1) which differ by atmost 1. A graph which follows prime cordial labeling is referred to as prime cordial graph [14]. $K_{1,m} * K_{1,n}$ is the graph obtained from $K_{1,m}$ by attaching root of a star $K_{1,n}$ at each pendant vertex of $K_{1,m}$. A Broom Graph $B_{n,m}$ is a graph with n vertices, consisting of a path P_m with m vertices and (n – m) pendant vertices attached to one end of the path.

3. Main results

Definition 3.1

Let G = (p, q) be a graph with p vertices and q edges. A SPC labeling of a graph G with vertex collection V(G) is a bijection α : V(G) → {0, 1, 2, …, p – 1} to the extent that its induced binary edge labeling function α* : E(G) → {0, 1} is defined by α*(uv) = $\begin{cases} 0 & \text{if } [((\alpha(u))^2, (\alpha(v))^2) + 2d(u,v)] \equiv 0 \pmod 2 \\ 1 & \text{if } [((\alpha(u))^2, (\alpha(v))^2) + 2d(u,v)] \equiv 1 \pmod 2 \end{cases}$. The difference between the edges designated as 0, denoted by $e_\alpha(0)$ and the edges designated as 1, denoted by e_α (1) is atmost 1. A graph which follows SPC labeling is referred to as SPC graph.

Lemma 3.2 (Characterization of edge parity)

Let α be a SPC labeling. Every edge e = uv gets the label 0 iff both α(u) ≡ 0 (mod 2) and α(v) ≡ 0 (mod 2). Consequently, e = uv gets the label 1 iff atleast one of the α(u) or α(v) is ≡ 1 (mod 2).

Proof:

By the definition of SPC, α*(uv) =

$$\begin{cases} 0 & \text{if } [((\alpha(u))^2, (\alpha(v))^2) + 2d(u,v)] \equiv 0 \pmod 2 \\ 1 & \text{if } [((\alpha(u))^2, (\alpha(v))^2) + 2d(u,v)] \equiv 1 \pmod 2 \end{cases}$$

By the number theoretical properties, $((\alpha(u))^2, ((\alpha(v))^2) = (\alpha(u), \alpha(v))^2$

Case(i)

Let α(u) ≡ 0 (mod 2) and α(v) ≡ 0 (mod 2) ⇔ (α(u), (α(v)) ≡ 0 (mod 2)

⇔ $(\alpha(u), (\alpha(v))^2 \equiv 0 \pmod 2$ ⇔ $((\alpha(u))^2, ((\alpha(v))^2) \equiv 0 \pmod 2$

⇔ $[((\alpha(u))^2, ((\alpha(v))^2)+2d(u,v)] \equiv 0 \pmod 2$ ⇔ α*(uv) = 0

Case(ii)

Let, at least α(u) ≡ 1 (mod 2) or v(v) ≡ 1 (mod 2) ⇔ (α(u), (α(v)) ≡ 1 (mod 2)

⇔ (α(u), $(α(v))^2$ ≡ 1 (mod 2) ⇔ ($(α(u))^2$, ($(α(v))^2$) ≡ 1 (mod 2)

⇔ [($(α(u))^2$, ($(α(v))^2$)+2d(u,v)] ≡ 1 (mod 2) ⇔ α*(uv) = 1

Hence, every edge e = uv gets the label 0 if both α(u) ≡ 0 (mod 2) and α (v) ≡ 0 (mod 2). Consequently, e = uv gets the label 1 iff atleast one of the α(u) or α(v) is ≡ 1 (mod 2).

Theorem 3.3

Let H be an arbitrary finite simple graph. For any integer k ≥ 1, the graph G ≅ 2kH is a SPC graph.

Proof:

Let H be any arbitrary finite simple graph with vertex set V(H) and edge set E(H) such that, |V(H)| = n, |E(H)| = m

Let G ≅ 2kH be the disjoint union of 2k copies of H with vertex set V(G) and edge set E(G) such that, |V(G)| = 2kn, |E(G)| = 2km

To prove that G is a SPC graph

Let G = $\cup_{i=1}^{2k} H_i = \{\cup_{i=1}^{k} H_{2i-1}\} \cup \{\cup_{i=1}^{k} H_{2i}\} = G_0 \cup G_1$

where $G_0 = \{\cup_{i=1}^{k} H_{2i-1}\}$ and $G_1 = \{\cup_{i=1}^{k} H_{2i}\}$

We get, $|V(G_0)| = kn = |V(G_1)|$ and $|E(G_0)| = km = |E(G_1)|$

Define α : V(G) → $\mathcal{L}$ where $\mathcal{L}$ = {0, 1, …, 2kn – 1}

$\mathcal{L}$ = {x ∈ $\mathcal{L}$: x ≡ 0 (mod 2)} ∪{x ∈ $\mathcal{L}$: x ≡ 1 (mod 2)}

$\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$ where $\mathcal{L}_1$ = {x ∈ $\mathcal{L}$: x ≡ 0 (mod 2)} and $\mathcal{L}_2$ = {x ∈ $\mathcal{L}$: x ≡ 1 (mod 2)}

⇒ $|\mathcal{L}_1| = kn = |\mathcal{L}_2|$

Now, label the vertices of the graphs in G_0 and G_1 bijectively with the element in $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively.

Define α*: E(G) → ε where ε = {0, 1}

Consider an edge e = uv such that,

$$\alpha^*(uv) = \begin{cases} 0 & \text{if} \quad [((\alpha(u))^2, (\alpha(v))^2) + 2d(u,v)] \equiv 0 \ (\text{mod } 2) \\ 1 & \text{if} \quad [((\alpha(u))^2, (\alpha(v))^2) + 2d(u,v)] \equiv 1 \ (\text{mod } 2) \end{cases}$$

By lemma 3.2, the induced edge labels of G_0 and G_1 are labeled as 0 and 1 respectively.

$|E(G_0)| = km \Rightarrow e_\alpha(0) = km$

$|E(G_1)| = km \Rightarrow e_\alpha(1) = km$

$|e_\alpha(0) - e_\alpha(1)| = |km - km| = 0 \leq 1$

Hence, For any integer $k \geq 1$, the graph $G \cong 2kH$ is a SPC graph.

Theorem 3.4

Every Broom graph $B_{n,m}$ is a SPC graph

Proof:

Let G be a Broom graph $B_{n,m}$ where n is the total number of vertices of G and m is the number of vertices of path.

The vertex set V(G) and edge set E(G) are defined by

$V(G) = \{u_i : 1 \leq i \leq n\}$; $E(G) = \{u_i u_{i+1} : 1 \leq i \leq m - 1\} \cup \{u_m u_{m+i} : 1 \leq i \leq n - m\}$

Then, G is of order n and size n – 1

There are two cases where $n \equiv 0 \ (\text{mod } 2)$ and $n \equiv 1 \ (\text{mod } 2)$

Case 1: $n \equiv 0$ (mod 2)

The function $\alpha : V(G) \rightarrow \{0, 1, 2, \ldots, n - 1\}$ is defined by

$$\alpha(u_i) = \begin{cases} 2i - 2 & ; \quad 1 \leq i \leq \frac{n}{2} \\ 2i - n - 1 & ; \quad \frac{n+2}{2} \leq i \leq n \end{cases}$$

Then, the induced edge labels are defined by

$$\alpha^*(uv) = \begin{cases} 0 & \text{if} \quad [((\alpha(u))^2, (\alpha(v))^2) + 2d(u,v)] \equiv 0 \ (\text{mod } 2) \\ 1 & \text{if} \quad [((\alpha(u))^2, (\alpha(v))^2) + 2d(u,v)] \equiv 1 \ (\text{mod } 2) \end{cases}$$

There are three subcases where $n < 2m$, $n > 2m$ and $n = 2m$

Subcase (i): $n < 2m$

By lemma 3.2, the induced edge labels are $\alpha^*(u_m u_{m+i}) = 1 : 1 \leq i \leq n - m$;

$$\alpha^*(u_i u_{i+1}) = \begin{cases} 0 & ; \quad 1 \le i \le \frac{n-2}{2} \\ 1 & ; \quad \frac{n}{2} \le i \le m-1 \end{cases}$$

We observe that, $e_\alpha(0) = \frac{n-2}{2}$. $e_\alpha(1) = \frac{n}{2}$

Thus, $|e_\alpha(0) - e_\alpha(1)| \le 1$

Subcase (ii): n >2m

By lemma 3.2, the induced edge labels are $\alpha^*(u_i u_{i+1}) = 0 : 1 \le i \le m - 1$;

$$\alpha^*(u_m u_{m+i}) = \begin{cases} 0 & ; \quad 1 \le i \le \frac{n-2m}{2} \\ 1 & ; \quad \frac{n-2m+2}{2} \le i \le n-m \end{cases}$$

We observe that, $e_\alpha(0) = \frac{n-2}{2}$; $e_\alpha(1) = \frac{n}{2}$

Thus, $|e_\alpha(0) - e_\alpha(1)| \le 1$

Subcase (iii): n = 2m

By lemma 3.2, the induced edge labels are

$\alpha^*(u_i u_{i+1}) = 0 : 1 \le i \le m - 1$; $\alpha^*(u_m u_{m+i}) = 1 : 1 \le i \le n - m$

We observe that, $e_\alpha(0) = m - 1$; $e_\alpha(1) = m$

Thus, $|e_\alpha(0) - e_\alpha(1)| \le 1$

Case 2: n ≡ 1 (mod 2)

The function $\alpha : V(G) \to \{0, 1, 2, \dots, n - 1\}$ is defined by

$$\alpha(u_i) = \begin{cases} 2i-2 & ; \quad 1 \le i \le \frac{n+1}{2} \\ 2i-n-2 & ; \quad \frac{n+3}{2} \le i \le n \end{cases}$$

There are three subcases where $n < 2m - 1$, $n > 2m - 1$ and $n = 2m - 1$

Subcase (i): n < 2m – 1

By lemma 3.2, the induced edge labels are $\alpha^*(u_m u_{m+i}) = 1 : 1 \le i \le n - m$;

$$\alpha^*(u_i u_{i+1}) = \begin{cases} 0 & ; \quad 1 \le i \le \frac{n-1}{2} \\ 1 & ; \quad \frac{n+1}{2} \le i \le m-1 \end{cases}$$

We observe that, $e_\alpha(0) = \frac{n-1}{2}$; $e_\alpha(1) = \frac{n-1}{2}$

Thus, $|e_\alpha(0) - e_\alpha(1)| \le 1$

Subcase (ii): n > 2m – 1

By lemma 3.2, the induced edge labels are $\alpha^*(u_i u_{i+1}) = 0 : 1 \le i \le m-1$;

$$\alpha^*(u_m u_{m+i}) = \begin{cases} 0 \quad ; & 1 \le i \le \frac{n-2m+1}{2} \\ 1 \quad ; & \frac{n-2m+3}{2} \le i \le n-m \end{cases}$$

We observe that, $e_\alpha(0) = \frac{n-1}{2}$ $e_\alpha(1) = \frac{n-1}{2}$

Thus, $|e_\alpha(0) - e_\alpha(1)| \le 1$

Subcase (iii): n = 2m – 1

By lemma 3.2, the induced edge labels are

$\alpha^*(u_i u_{i+1}) = 0 : 1 \le i \le m-1$; $\alpha^*(u_m u_{m+i}) = 1 : 1 \le i \le n-m$

We observe that, $e_\alpha(0) = m-1$; $e_\alpha(1) = m-1$

Thus, $|e_\alpha(0) - e_\alpha(1)| \le 1$

In both cases, Broom graph $B_{n,m}$ admits square prime cordial labeling

Hence, Broom graph $B_{n,m}$ is a square prime cordial graph

Example 3.5 (When n ≡ 1 (mod 2) and n > 2m – 1)

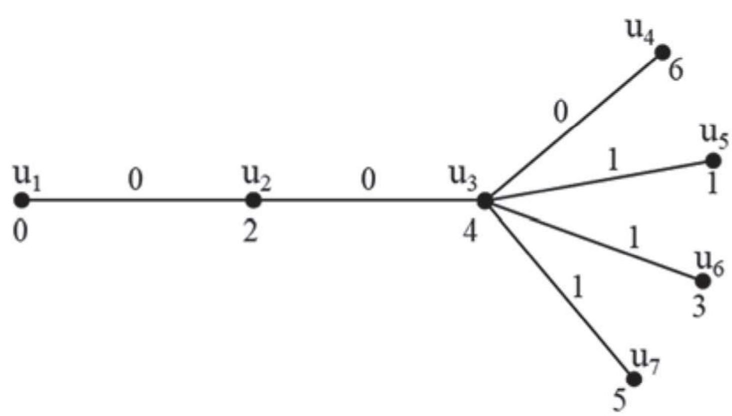

Figure 3.1: ($B_{7,3}$)

Theorem 3.6

The graph $K_{1,m} * K_{1,n}$ is a Square Prime Cordial graph

Proof:

Let G be a $K_{1,m} * K_{1,n}$ graph

The vertex set V(G) and edge set E(G) are defined by

$V(G) = \{w\} \cup \{u_i : 1 \le i \le m\} \cup \{u_{ij} : 1 \le i \le m, 1 \le j \le n\}$;

$E(G) = \{u_i w : 1 \le i \le m\} \cup \{u_i u_{ij} : 1 \le i \le m, 1 \le j \le n\}$

Then, G is of order mn + m + 1 and size m(n + 1)

There are two cases where $m \equiv 0 \pmod 2$ and $m \equiv 1 \pmod 2$

Case (i): m ≡ 0 (mod 2)

The function $\alpha: V(G) \to \{0, 1, 2, \ldots, m(n+1)\}$ is defined by

$\alpha(w) = m(n+1)$; $\alpha(u_i) = i - 1 : 1 \le i \le m$; $\alpha(u_{ij}) = jm - 1 + i : 1 \le i \le m$, $1 \le j \le n$

Then, the induced edge labels are defined by

$$\alpha^*(uv) = \begin{cases} 0 & \text{if } [((\alpha(u))^2, (\alpha(v))^2) + 2d(u,v)] \equiv 0 \pmod 2 \\ 1 & \text{if } [((\alpha(u))^2, (\alpha(v))^2) + 2d(u,v)] \equiv 1 \pmod 2 \end{cases}$$

By lemma 3.2, the induced edge labels are given by

$$\alpha^*(u_i w) = \begin{cases} 1 & \text{if } i \equiv 0 \pmod 2 \\ 0 & \text{if } i \equiv 1 \pmod 2 \end{cases} : 1 \le i \le m;$$

$$\alpha^*(u_i u_{ij}) = \begin{cases} 1 & \text{if } i \equiv 0 \pmod 2 \\ 0 & \text{if } i \equiv 1 \pmod 2 \end{cases} : 1 \le i \le m, 1 \le j \le n$$

We observe that, $e_\alpha(0) = \frac{m(n+1)}{2}$; $e_\alpha(1) = \frac{m(n+1)}{2}$

Case (ii): m ≡ 1 (mod 2)

There are three subcases, where n = 2, $n \equiv 0 \pmod 2$ & (n > 2) and $n \equiv 1 \pmod 2$

Subcase (i): n = 2

The function $\alpha : V(G) \to \{0, 1, 2, \ldots, 3m\}$ is defined by

$\alpha(w) = 3m - 1$; $\alpha(u_i) = i - 1 : 1 \le i \le m$; $\alpha(u_{m1}) = m$;

$\alpha(u_{ij}) = jm - (j - 1) + i : 1 \le i \le m - 1, j = 1, 2$; $\alpha(u_{m2}) = 3m$

By lemma 3.2, the induced edge labels are $\alpha^*(u_i w) = \begin{cases} 1 & \text{if } i \equiv 0 \pmod 2 \\ 0 & \text{if } i \equiv 1 \pmod 2 \end{cases}$:
$1 \le i \le m$;

$\alpha^*(u_i u_{ij}) = \begin{cases} 1 & \text{if } i \equiv 0 \pmod 2 \\ 0 & \text{if } i \equiv 1 \pmod 2 \end{cases} : 1 \le i \le m, j = 1, 2$

We observe that, $e_\alpha(0) = \frac{3m-1}{2}$; $e_\alpha(1) = \frac{3m+1}{2}$

Subcase (ii): n ≡ 0 (mod 2) and (n > 2)

The function $\alpha : V(G) \to \{0, 1, 2, \ldots, m(n + 1)\}$ is defined by

$\alpha(w) = m(n + 1) - 1$; $\alpha(u_i) = i - 1 : 1 \le i \le m$; $\alpha(u_{m1}) = m$; $\alpha(u_{mn}) = m(n + 1)$;

$\alpha(u_{ij}) = jm - (j - 1) + i : 1 \le i \le m - 1, 1 \le j \le n$; $\alpha(u_{mj}) = (n + 1)(m - 1) + j : 2 \le j \le n - 1$

By lemma 3.2, the induced edge labels are given by

$\alpha^*(u_i w) = \begin{cases} 1 & \text{if } i \equiv 0 \pmod 2 \\ 0 & \text{if } i \equiv 1 \pmod 2 \end{cases} : 1 \le i \le m;$

$\alpha^*(u_i u_{ij}) = \begin{cases} 1 & \text{if } i \equiv 0 \pmod 2 \\ 0 & \text{if } i \equiv 1 \pmod 2 \end{cases} : 1 \le i \le m - 1, 1 \le j \le n;$

$\alpha^*(u_m u_{mj}) = \begin{cases} 1 & \text{if } j \equiv 1 \pmod 2 \\ 0 & \text{if } j \equiv 0 \pmod 2 \end{cases} : 1 \le j \le n - 1;\ \alpha^*(u_m u_{mn}) = 1$

We observe that, $e_\alpha(0) = \frac{m(n+1)-1}{2}$; $e_\alpha(1) = \frac{m(n+1)+1}{2}$

Subcase (iii): n ≡ 1 (mod 2)

The function $\alpha : V(G) \to \{0, 1, 2, \ldots, m(n + 1)\}$ is defined by

$\alpha(w) = m(n + 1)$; $\alpha(u_i) = i - 1 : 1 \le i \le m$; $\alpha(u_{m1}) = m$;

$\alpha(u_{ij}) = jm - (j - 1) + i : 1 \le i \le m - 1, 1 \le j \le n$; $\alpha(u_{mj}) = (n + 1)(m - 1) + j : 2 \le j \le n$

By lemma 3.2, the induced edge labels are given by

$\alpha^*(u_i w) = \begin{cases} 1 & \text{if } i \equiv 0 \pmod 2 \\ 0 & \text{if } i \equiv 1 \pmod 2 \end{cases} : 1 \le i \le m;$

$\alpha^*(u_m u_{mj}) = \begin{cases} 1 & \text{if } j \equiv 1 \pmod 2 \\ 0 & \text{if } j \equiv 0 \pmod 2 \end{cases} : 1 \le j \le n$

$\alpha^*(u_i u_{ij}) = \begin{cases} 1 & \text{if } i \equiv 0 \pmod 2 \\ 0 & \text{if } i \equiv 1 \pmod 2 \end{cases} : 1 \le i \le m - 1, 1 \le j \le n;$

We observe that, $e_\alpha(0) = \frac{m(n+1)}{2}$; $e_\alpha(1) = \frac{m(n+1)}{2}$

In both cases, $|e_\alpha(0) - e_\alpha(1)| \le 1$

$K_{1,m} * K_{1,n}$ admits Square Prime Cordial labeling

Hence, $K_{1,m} * K_{1,n}$ is a Square Prime Cordial graph

Example 3.7

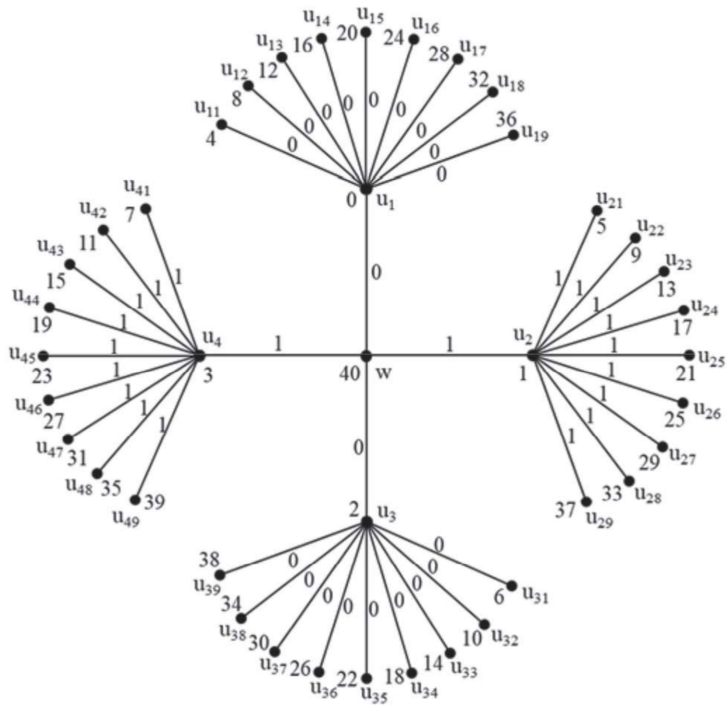

Figure 3.2: ($K_{1,4} * K_{1,9}$)

Conclusion

In this paper, SPC labeling for the broom graph and $K_{1,m} * K_{1,n}$ and square prime cordiality of disjoint union of 2k copies of any graph have been discussed. This study can be further extended to the disjoint union of 2k + 1 copies and some other results. Some cycle related graphs are not SPC as they do not meet the cardinality condition. This can be investigated in further studies.

Conflict of Interest

The authors hereby declare no potential conflicts of interest with respect to the research, funding, authorship, and/or publication of this article

References

[1]. J. A. Gallian, A dynamic survey of graph labeling. The Electronics Journal of Combinatornics, (2024), # Ds6, 1-712.

[2]. F. Harary Graph Theory Narosa Publishing House, New Delhi, 1988.

[3]. S Dhanalakshmi and N. Parvathi 2019 J. Phys.: Conf. Ser. 1377 012027. DOI:10.1088/1742-6596/1377/1/012027.

[4]. Parul B Pandya, N P Shrimali, Vertex-Edge Neighborhood Prime Labeling of Some Graphs., Volume 7, Issue 10, 2018, ISSN NO:2279-543X, Page No. 735-743.

[5]. Arockiam, Lourdusamy & F, Patrick. (2016). Sum divisor cordial graphs. Proyecciones. 35. 119-136. 10.4067/S0716-09172016000100008.

[6]. S. Stanly, I. Gnanaselvi, S.Alice Pappa, Mean Square Difference Cordial Labeling of Path Related Graphs, International Journal of Food and Nutritional Sciences, ISSN: 2320 7876/ 2022, Issue: 11, Pages: 16554 – 16573.

[7]. S. Stanly, I. Gnanaselvi, S. Alice Pappa, Mean Square Difference Cordial Labeling of Some Graphs, Design Engineering, ISSN: 0011-9342/ year 2021, Issue: 9, Pages: 5108-5115.

[8]. S. Dhanalakshmi and N Parvathi, (2018) *Journal of Physics Conference Series* 2516, 210013 (2022). DOI: 10.1063/5.0109630

[9]. Dr. A. Sugumaran and P. Vishnu Prakash. Some new results of Prime Cordial Labeling. International J. of Pure & Engg. Mathematics (IJPEM) ISSN 2348-3881, Vol. 5 No. I (April, 2017), pp. 9-14.

[10]. M. Sundaram, R. Ponraj and S. Somasundaram, Prime cordial labeling of graphs, Journal of Indian Academy of Mathematics, 27(2005) 373-390.

[11]. I. Cahit, Cordial graph A weaker version of graceful and harmonious graphs, Ars Combinatoria, 23(1987), 201-207.

[12]. A. Rosa, (1967) On Certain Valuations of the vertices of a graph, in Theory of Graphs, International Symposium, Rome, July 1966, Gordon and Breach, New York and Dunod, Paris, pp.349-355.

[13]. Ru, Min. "Greatest Common Divisors." University of Houston, spring 2011.

[14]. K. Gayathri, A. Sasikala, C. Sekar, (2021) *Journal of Physics Conference Series* 1947 012016. DOI: 10.1088/1742-6596/1947/1/012016