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Optimization of Single-Seller Multi-item in Multiple Outlets Distribution Network Fuzzy Inventory System with Lead Time and Carbon Emission Cost

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Abstract

Managing multiple products across various locations reduces total costs through consolidated replenishment. This strategy reduces ordering costs because fewer transactions result in better pricing. It also reduces shipping costs by using combined shipments to minimize the cost of transportation. However, in practical environments, important parameters of inventory, such as demand, cost and lead time are often uncertain resulting in inaccuracies in cost estimation. To represent the uncertainty involved in the model, hexagonal fuzzy numbers are used. The fuzzy objective function has computational difficulties due to its specific structure. The model is solved using the alpha cut technique combined with the Lagrangian method, while the fuzzy counterparts of the remaining constraints are available. An algorithm is developed to determine the optimal order quantity for each item at every outlet. The proposed approach simultaneously minimizes the total cost of the entire inventory system. An evaluation of fuzzy multi-item in a multi-outlet distribution network inventory system is presented, along with a comparison with the crisp multi-item in the same system, using numerical illustrations. Finally, graphical representations of the proposed system's performance are provided. This fuzzy inventory framework is effective in optimizing results for multi-items across various outlets.

Keywords: Fuzzy multiple items in multiple outlets distribution network inventory system; Alpha cut procedure; Optimum order size; Least joined entire cost; Lagrangian process.

2010 AMS subject classification: 94D05, 35Q93.

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1. Introduction

The two-echelon inventory system, comprising a single seller and multiple outlets distribution network for multiple items, represents a significant area of research. This structure is highly applicable to various real-world scenarios within supply chain coordination. A two-fold inventory system has been extended to multiple outlets distribution network methodology. The synchronization technique and organisational formation stream are important in the analysis of inventory by multiple outlets distribution network. The model is based on replenishment of the inventory from the unique seller to several outlets that may have different demand patterns. It aims at proper stocking and distribution of products through multiple outlets distribution network, and reducing overall costs. It is ensured that stock placement in varied outlets is enhanced aiming at making the products easily accessible for customers, while cutting down on various costs related to storage and distribution. The proposed framework coordinates inventory and distribution for diverse product lines across multiple outlets distribution network, balancing economic objectives with environmental sustainability.

The paper is organised as: In section 2, the literature survey is presented. Section 3 outlines the mathematical notations and the specific assumptions required to formulate the inventory system. Section 4 presents the crisp formulation of the inventory problem, focusing on the simultaneous optimization of the joint total cost function and the economic order quantities. Furthermore, alpha cut technique and Extension of Lagrangian process are utilized to solve the fuzzy system. To reach the most efficient outcome, a fuzzy inventory system is established, supported by an algorithm that streamlines the optimization of the objective function. Section 5 utilizes numerical examples and pictorial figures to validate the performance of the proposed multi-item, multi-outlet distribution network under both deterministic and fuzzy conditions. Section 6 presents a comparative analysis of the results, while Section 7 provides concluding remarks and suggests directions for future research.

2. Literature Survey

Most of the inventory types that determine optimal tactics are designed at sole item with single outlet. Managing stock for a single item at one location using a set policy typically leaves the company's inventory costs and profit margins unchanged. For instance, substitutes to a sole item, several companies or customers or initiatives are stimulated to stock several items with multiple outlets distribution network in their industrial unit for additional beneficial marketable locations. In addition, it attracts the buyers to procure more than a few items. Multiple consumers, multiple traders and multiple items stream sequence in which every consumer and every wholesaler is limited by storehouse restriction in gathering the items [18]. Single-seller with multi-retailer was affirmed in [4] which the sector collection optimization and substantial mechanisms are recognized to adopt the prototype. An integrated supply system featuring a single source point and multiple consumer channels across a multi-item inventory [20]. Furthermore, [14] considers complex variables such as random ordering, multiple objects, and various customer profiles. In practice, there is no guarantee that manufactured objects will be defect-free. An evaluation of a two-tier supply model, consisting of one manufacturer and one retailer, to determine its potential for commercial viability [24].

An inventory system [16] using principal warehouse and multiple trade outlets, working together during deficiency by moving inventory between them. The investigation focuses on the circumstance of three outlets (carrying sites), which encapsulates maximum of the features and transactions in multi-site organizations with widespread merging. A model with multi-items in which demands of the items are dependent on time and cost power [13]. Multi-product, single-level, multi-period inventory model, which has been demonstrated by an undefined economic situation [6]. The ultimate results are obtained through the search method. A single seller with multiple traders' management [10] in extremely forceful and free advertisements. Multiple items multiple periods inventory model with identified-deterministic flexible demands under a restricted obtainable budget [11]. Stochastic demand in real-world supply chain difficulties; they are infrequently measured in the scrutiny of supply chain arrangements, particularly the single producer multiple vendors supply chain arrangements [23]. Due to the dimension hurdle, the fast Fourier transform method is hosted, which considerably diminishes the computational difficulty of resolving the supply chain system. A non-rapid category of items using a novel decision-constructing optimum refill strategy pattern model with multiitem multi ware houses [5].

This model initiates two separate supply configurations at the start of the lead period. It balances these against specific facility volume constraints and integrates transportation efficiencies to reduce overall expenditure [15]. Optimizes a combined inventory framework by linking ordering cost reductions to lead-time variables, all while employing trapezoidal fuzzy logic to manage stochastic fluctuations [21]. By adopting a pentagonal fuzzy framework, this study analyses a two-stage production environment characterized by imprecise parameters and decision variables [22]. A multi-item inventory system is proposed to manage claim-dependent stock levels, utilizing fuzzy set theory to represent the uncertainty of the variables involved. Articles are getting worse at a stable rate and are retailed through various exits as per the system [7]. A combined seller–purchaser supply sequence type of backorder amount deduction, cost-related demand and carbon discharge rate [9].

Multi-item unremitting assessment inventory system and indeterminate request, eminence enhancement, structure rate drop in addition to disparity resistor in principal period [8]. An eco-friendly inventory system characterized by product degradation and imperfect manufacturing processes. The framework is designed to optimize stock levels while strictly monitoring carbon emission constraints [19]. The complexities of decision-making under uncertainty, this study employs a Lagrangian relaxation technique, a method widely validated within the field of operations research for solving constrained optimization problems [17]. In contrast to traditional inventory frameworks [12], this approach yields a more cost-effective solution while simultaneously compressing lead times to improve operational efficiency.

The application of fuzzy set theory [25] provides a robust framework for managing imprecise data clusters within the field of operations research. Hexagonal fuzzy number is utilised as a respected logic to comfort imprecision evidence [1]. Arithmetic processes in fuzzy numbers through the utility code [2]. The manufacture inventory type optimized using serviceable imperfect goods by fuzzy or crisp manufacture price amount is assessed by Chen et al. [3].

3. Notations and Assumptions

The following symbols and mathematical notations are established to provide a consistent framework for this inventory model.

3.1. Notations

For the $s-$th outlet and $g-$th item are employed to construct the system.

$G -$ Number of items,

$O -$ Number of outlets,

$Q_{gs} -$ Order size for $g-$th item for $s-$th outlet,

$L_{gs} -$ Lead time span for $g-$th item of $s-$th outlet,

$A_{gs} -$ Ordering cost for $g-$th item of $s-$th outlet per order,

$m_{gs} -$ Lots size for $g-$th item, its factory-made properties are provided from the seller to $s-$th outlet phase,

$D_{gs} -$ Average demand per unit time for $g-$th item of $s-$th outlet,

$P_{gs} -$ Production rate for $g-$th item of the $s-$th outlet of the seller $P_{gs} > D_{gs}$,

$S_{gs} -$ Seller's setup cost per arrangement for $g-$th item of $s-$th outlet,

C_{vgs} — Production cost for $g-$th item of $s-$th outlet funded through seller $C_{vgs} > C_{bgs}$,

C_{bgs} — Purchasing cost for $g-$th item of $s-$th outlet backed by the outlet,

r_{gs} — Annually inventory holding price for $g-$th item for $s-$th outlet in which every dollar is exploited in frameworks,

R_{gs} — Reorder level for $g-$th item of the $s-$th outlet,

VEC_{vgs} — Seller's flexible carbon emission cost for $g-$th item of $s-$th outlet,

FEC_{vgs} — Seller's stable carbon emission cost for $g-$th item of $s-$th outlet,

FTC_{vgs} — Seller's stable transportation cost for $g-$th item of $s-$th outlet,

VTC_{vgs} — Seller's flexible transportation price for $g-$ th item of $s-$th outlet,

$ITCMO$ — Total cost of the entire crisp inventory system,

$I\tilde{T}CMO$ — Total cost of the entire fuzzy inventory system.

3.2. Assumptions

1. The coordination comprises single-seller with multiple outlets distribution network designed at multiple items inventory system.
2. The model is developed for G number of items, O number of outlets and g_s items are sold. So, $\sum_{s=1}^{O} t_s = G$.
3. The demand of the outlets is decided on their own.
4. The purchaser orders for $g-$th item a lot of size Q_{gs} in $s-$th outlet and the seller makings $m_{gs} Q_{gs}$ with a limited manufacture ratio $P_{gs}(P_{gs} > D_{gs})$. At a single setup, size Q_{gs} is transported to the outlet for m_{gs} times. The seller's set-up cost is S_{gs} for every manufacturing run. The consumer pays an ordering price A_{gs} for every order size Q_{gs}
5. The mandate of a $g-$th item for $s-$th outlet X_{gs} throughout lead time L_{gs} tracks a normal distribution through mean $\mu_{gs} L_{gs}$ and standard deviation $\sigma_{gs}\sqrt{L_{gs}}$
6. The stock is uninterruptedly observed. Upon reaching the reorder point R_{gs}, the outlet has to order the items.
7. The reorder point is equal to the total of expected demand for the time of safety stock and principal time for $g-$th item along $s-$th outlet + safety stock, $R_{gs} = D_{gs}L_{gs} + k_{gs}\sigma_{gs}\sqrt{L}_{gs}$ where L_{gs} is the safety factor.

8. The lead time L_{gs} for all items is like and it consists of n_{gs} commonly self-determining segments. The $z-$th segment has a normal distribution b_{gsz}, minimum duration a_{gsz} and crashing price each unit period c_{gsz}. For appropriateness, c_{gsz} is systematized as $c_{gs1} < c_{gs2} < c_{gs3} < \ldots < c_{gsn}$.
9. The modules of lead time change one at a time, beginning with the first module. Because the initial module is selected based on the lowest crashing cost per unit, all subsequent modules are prioritized using the same cost-efficiency logic.
10. Let $L_{gs0}=\sum_{z=1}^{n} b_{gsz}$ and L_{gsz} be the duration of principal time consuming segments 1, 2, 3,...., z_{gsp} which crash to their lowest time, then L_{gsz} can be denoted as $L_{gsz}=L_{gs0}-\sum_{z=1}^{n_{gs}}(b_{gsz}-a_{gsz}), z=1,2,3,\ldots,n_{gs}$; and the principal period crashing price per set $R(L_{gs})$ is given as $R(L_{gs})=c_{gsz}(L_{gs(z-1)}-L_{gs})+\sum_{j=1}^{z-1} c_{gsj}(L_{gs(j-1)}-L_{gs})$ $L_{gs}\in[L_{gsz}, L_{gs(z-1)}]$ Lead time duration corresponds directly to the full length of each transport cycle. Consequently, the costs associated with accelerating these timelines are incurred during every delivery rotation, reflecting a persistent connection between lead time and crashing expenses.
11. The decrease of lead time L_{gsz} with condensed ordering price A_{gs} and A_{gs} is resolutely the concave function of L_{gsz} i.e., $A_{gs}'(L_{gsz})>0$ and $A_{gs}''(L_{gsz})<0$.
12. To maintain the model's balance, it is necessary to shift all sustained seller charges to the outlet following the implementation of a reduced lead time.

4. Model Formulation

This study establishes an inventory model for a single-seller, multi-outlet distribution network, accounting for multi-item sets and carbon emission costs. The framework is analysed under both crisp and fuzzy environments to evaluate lead time impacts.

4.1. Crisp Multi- item in Multi-outlet Distribution Network Inventory System

Total price of multi- item in multi-outlet distribution network (ITCMO)

Total price per unit time is derived for multi-item in multi-outlet distribution network and summated for resulting modules.

Ordering price for $g-$th item of $s-$th outlet for each unit time

$$=\frac{A_{gs}}{Q_{gs}/D_{gs}}=\frac{A_{gs}D_{gs}}{Q_{gs}} \tag{1}$$

Buyer's holding price for $g-$th item of $s-$th outlet for each unit time

$$=\left(\frac{Q_{gs}}{2}+k_{gs}\sigma_{gs}\sqrt{L_{gs}}\right)r_{gs}C_{bgs} \tag{2}$$

Lead time crashing price for $g-$th item of $s-$th outlet per unit time

$$=\left(\frac{D_{gs}}{Q_{gs}}\right)R(L_{gs}), \tag{3}$$

Seller setup cost for $g-$th item of $s-$th outlet for each year

$$=\left(\frac{D_{gs}}{m_{gs}Q_{gs}}\right)S_{gs}, \tag{4}$$

Seller's average inventory price for $g-$th item of s—th outlet

$$=\left\{\left[m_{gs}Q_{gs}\left(\frac{Q_{gs}}{P_{gs}}+(m_{gs}-1)\frac{Q_{gs}}{D_{gs}}\right)-\frac{m_{gs}{}^2Q_{gs}{}^2}{2P_{gs}}\right]-\left[\frac{Q_{gs}{}^2}{D_{gs}}(1+2+......+(m_{gs}-1))\right]\right\}\frac{D_{gs}}{m_{gs}Q_{gs}}$$

$$=\frac{Q_{gs}}{2}\left[m_{gs}\left(1-\frac{D_{gs}}{P_{gs}}\right)-1+\frac{2D_{gs}}{P_{gs}}\right].$$

So, the seller's holding price for $g-$th item of $s-$th outlet for each unit time

$$=\frac{Q_{gs}}{2}\left[m_{gs}\left(1-\frac{D_{gs}}{P_{gs}}\right)-1+\frac{2D_{gs}}{P_{gs}}\right]r_{gs}C_{vgs}. \tag{5}$$

Seller annual transportation cost for $g-$th item of $s-$th outlet

$$= m_{gs}(FTC_{vgs}+VTC_{vgs}), \tag{6}$$

Annual carbon emission cost for $g-$th item of $s-$th outlet

$$= m_{gs}FEC_{vgs}+Q_{gs}VEC_{vgs}. \tag{7}$$

Based on the assumptions, the joined entire price per unit time for $g-$th item in $s-$th outlet is a collection of above stated costs specified as

$$\begin{aligned}ITCMO_{gs}(Q_{gs},L_{gs},m_{gs})=&\left[\frac{D_{gs}}{Q_{gs}}\left(A_{gs}+\frac{S_{gs}}{m_{gs}}+R(L_{gs})\right)-\frac{Q_{gs}r_{gs}C_{vgs}}{2}\left(\frac{m_{gs}D_{gs}}{P_{gs}}+1\right)+\frac{Q_{gs}r_{gs}}{2}\left(\left(m_{gs}+\frac{2D_{gs}}{P_{gs}}\right)\right.\right.\\&\left.\left.C_{vgs}+C_{bgs}\right)+r_{gs}C_{bgs}k_{gs}\sigma_{gs}\sqrt{L_{gs}}+Q_{gs}VEC_{vgs}+m_{gs}(FTC_{vgs}+VTC_{vgs}+FEC_{vgs})\right].\end{aligned} \tag{8}$$

The crisp joined entire price of $s-$th outlet for entire G items is denoted as $ITCMO_s$, that is $ITCMO_s=\sum_{g=1}^{t_s} ITCMO_{gs}(Q_{gs},L_{gs},m_{gs})$ for g=1,2,....,t_s and the crisp joined entire price of the entire system is

$$ITCMO = \sum_{s=1}^{O} ITCMO_s = \sum_{s=1}^{O}\sum_{g=1}^{t_s} ITCMO_{gs}\,(Q_{gs}, L_{gs}, m_{gs}),$$

$$ITCMO = \sum_{s=1}^{O}\sum_{g=1}^{t_s}\left[\frac{D_{gs}}{Q_{gs}}\left(A_{gs}+\frac{S_{gs}}{m_{gs}}+R(L_{gs})\right)-\frac{Q_{gs}r_{gs}C_{vgs}}{2}\left(\frac{m_{gs}D_{gs}}{P_{gs}}+1\right)+\frac{Q_{gs}r_{gs}}{2}\left(\left(m_{gs}+\frac{2D_{gs}}{P_{gs}}\right)C_{vgs}+C_{bgs}\right)\right.$$
$$\left.+r_{gs}C_{bgs}k_{gs}\sigma_{gs}\sqrt{L_{gs}}+Q_{gs}VEC_{vgs}+m_{gs}(FTC_{vgs}+VTC_{vgs}+FEC_{vgs})\right]. \quad (9)$$

With the specific rate of m_{gs} and L_{gs} the joined entire price for $g-$th item of $s-$th outlet is $ITCMO_{gs}(Q_{gs},L_{gs},m_{gs})$, then optimum order size Q_{gs} is attained, while joined entire price$ITCMO_{gs}(Q_{gs},L_{gs},m_{gs})$ is minimum. In order to get minimization of $ITCMO_{gs}(Q_{gs},L_{gs},m_{gs})$ the partial derivative of $ITCMO_{gs}(Q_{gs},L_{gs},m_{gs})$ with Q_{gs} is found and equated to zero. Then,

$$-\frac{D_{gs}}{Q_{gs}^{\;2}}\left(A_{gs}+\frac{S_{gs}}{m_{gs}}+R(L_{gs})\right)-\frac{r_{gs}C_{vgs}}{2}\left(\frac{m_{gs}D_{gs}}{P_{gs}}+1\right)+\frac{r_{gs}}{2}\left(\left(m_{gs}+\frac{2D_{gs}}{P_{gs}}\right)C_{vgs}+C_{bgs}\right)+VEC_{vgs}=0. \quad (10)$$

For a static m_{gs}, and L_{gs}, the joined entire price for $g-$th item of $s-$th outlet $ITCMO_{gs}(Q_{gs},L_{gs},m_{gs})$is optimistic fixed in Q_{gs} While examining the appropriate conditions to get least price of $ITCMO_{gs}(Q_{gs},L_{gs},m_{gs})$ second order partial derivatives of $ITCMO_{gs}(Q_{gs},L_{gs},m_{gs})$ with respect to Q_{gs} are used to attain

$$\frac{\partial^2 ITCMO_{gs}(Q_{gs},L_{gs},m_{gs})}{\partial Q_{gs}^{\;2}}=\frac{2D_{gs}}{Q_{gs}^{\;3}}\left(A_{gs}+\frac{S_{gs}}{m_{gs}}+R(L_{gs})\right)>0. \quad (11)$$

Therefore, $ITCMO_{gs}(Q_{gs},L_{gs},m_{gs})$ is convex in Q_{gs} for a static m_{gs} and L_{gs}. As a result, optimum values $Q_{gs}{}^*$ decrease so that a local minimum is attained. Hence, the optimum order size Q_{gs} found by equation (10) is

$$Q_{gs}^{\;*}=Q_{gs}=\sqrt{\frac{2D_{gs}\left(A_{gs}+\frac{S_{gs}}{m_{gs}}+R(L_{gs})\right)}{r_{gs}\left(\left(m_{gs}\left(1-\frac{D_{gs}}{P_{gs}}\right)-1+\frac{2D_{gs}}{P_{gs}}\right)C_{vgs}+C_{bgs}\right)+2VEC_{vgs}}}. \quad (12)$$

4.2. Fuzzy Multi-item in Multi-outlet Distribution Network Inventory System

The fuzzy-joined entire price for multi-item in multi-outlet distribution network with a dual-echelon inventory structure framed. This study introduces an improved algorithm for determining optimum order quantities under uncertainty. The novelty of this approach is the integration of the (α)−cut technique with the Lagrangian method, which allows us to convert complex fuzzy goals into manageable intervals. While previous models often assume constant demand, costs and system related parameters, our model treats these as hexagonal fuzzy parameters. This solves the common problem of over-ordering or stockouts caused by unpredictable market changes.

4.2.1. Solution Methodology

4.2.1.1 Hexagonal fuzzy number

A hexagonal fuzzy number $\tilde{\zeta} = (\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6)$ is defined by the membership function as

$$\mu_{\tilde{\zeta}}(y) = \begin{cases} \dfrac{(y-\zeta_1)}{\zeta_2-\zeta_1}, & if\ \zeta_1 \le y \le \zeta_2 \\ \dfrac{(y-\zeta_2)}{\zeta_3-\zeta_2}, & if\ \zeta_2 \le y \le \zeta_3 \\ 1, & if\ \zeta_3 \le y \le \zeta_4 \\ \dfrac{(\zeta_4-y)}{\zeta_5-\zeta_4}, & if\ \zeta_4 \le y \le \zeta_5 \\ \dfrac{(\zeta_5-y)}{\zeta_6-\zeta_5}, & if\ \zeta_5 \le y \le \zeta_6 \\ 0, & otherwise \end{cases}.$$

Regarding hexagonal fuzzy number, the membership value is achieved for a precise range.

4.2.1.2 Hexagonal fuzzy number's arithmetic processes

Let $\tilde{\zeta} = (\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6)$ and $\tilde{\omega} = (\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6)$ be two hexagonal fuzzy numbers. Then

1. Sum of $\tilde{\zeta}$ and $\tilde{\omega}$ indicated in

   $\tilde{\zeta} \oplus \tilde{\omega} = (\zeta_1 + \omega_1, \zeta_2 + \omega_2, \zeta_3 + \omega_3, \zeta_4 + \omega_4, \zeta_5 + \omega_5, \zeta_6 + \omega_6)$ where $\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6$ and $\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6$ are any real numbers.

2. Product of $\tilde{\zeta}$ and $\tilde{\omega}$ indicated in

   $\tilde{\zeta} \otimes \tilde{\omega} = (\zeta_1\omega_1, \zeta_2\omega_2, \zeta_3\omega_3, \zeta_4\omega_4, \zeta_5\omega_5, \zeta_6\omega_6)$ where $\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6$

and $\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6$ remain any positive real numbers.

3. If $-\tilde{\omega} = (-\omega_6, -\omega_5, -\omega_4, -\omega_3, -\omega_2, -\omega_1)$ then the subtraction of $\tilde{\omega}$ from $\tilde{\zeta}$ is $\tilde{\zeta} \,!\, \tilde{\omega} = (\zeta_1 - \omega_6, \zeta_2 - \omega_5, \zeta_3 - \omega_4, \zeta_4 - \omega_3, \zeta_5 - \omega_2, \zeta_6 - \omega_1)$ where $\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6$ and $\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6$ are any real numbers.

4. Let $\frac{1}{\tilde{\omega}} = \tilde{\omega}^{-1} = \left(\frac{1}{\omega_6}, \frac{1}{\omega_5}, \frac{1}{\omega_4}, \frac{1}{\omega_3}, \frac{1}{\omega_2}, \frac{1}{\omega_1}\right)$ where $\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6$ are entirely real positive numbers, then the division of $\tilde{\zeta}$ and $\tilde{\omega}$ is $\tilde{\zeta} \oslash \tilde{\omega} = \left(\frac{\zeta_1}{\omega_6}, \frac{\zeta_2}{\omega_5}, \frac{\zeta_3}{\omega_4}, \frac{\zeta_4}{\omega_3}, \frac{\zeta_5}{\omega_2}, \frac{\zeta_6}{\omega_1}\right)$.

5. For any real number ξ, $\xi \otimes \tilde{\zeta} = \begin{cases} (\xi\zeta_1, \xi\zeta_2, \xi\zeta_3, \xi\zeta_4, \xi\zeta_5), \xi \geq 0 \\ (\xi\zeta_5, \xi\zeta_4, \xi\zeta_3, \xi\zeta_2, \xi\zeta_1), \xi < 0. \end{cases}$

4.2.1.3. Defuzzification for hexagonal Fuzzy Number

Defuzzification of the hexagonal fuzzy variable $\tilde{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6)$ is executed via the alpha (α)—cut technique, resulting in the crisp point estimate

$$P(\tilde{\alpha}) = \frac{\int_0^1 [L^{-1}(\alpha) + R^{-1}(\alpha)] d\alpha}{2} = \frac{\alpha_1 + \alpha_2 + \alpha_5 + \alpha_6}{4}. \quad (13)$$

4.2.1.4 Extension of Lagrangian method

Solving of the nonlinear programming problems [17] with equality constraints is possible with the help of the Lagrangian Method which can also be used to find an optimum solution for the inequality constraints. The core principle of the Lagrangian approach is that when an unconstrained maximum fails to meet the required criteria, the optimum solution under constraints will inevitably lie on the boundary of the feasible region.

The problem is

Minimize $y=f(x)$ subject to $g_s(x) \geq 0$, s = 1,2,3,.......,p.
The non-negativity constraints $x \geq 0$ (if any) are included in the m constraints.

The Lagrangian function is given by $L(x_t,\lambda_s) = f(x_t) - \sum_{s=1}^{p}\lambda_s g_s(x)$, where λ_s are Lagrange Multipliers given the constraints $g_s(x) \geq 0$.

Taking the partial derivatives of L with respect to x_s and λ_s, we obtain

$$\frac{\partial L}{\partial x_t} = \nabla f(x_t) - \sum_{s=1}^{p}\lambda_s \nabla g_s(x_t) = 0,\ s = 1,2,3,....,\mathrm{p}, t = 1,2,3,....,\mathrm{q}.$$

$$\frac{\partial L}{\partial \lambda_s} = -\sum_{s=1}^{p} g_s(x) = 0,\ s = 1,2,....\mathrm{p}.$$

Then, the technique of Extension of the Lagrangian method consists of the following steps.

Step 1. Solve the unconstrained problem Minimize $y=f(x_t)$. If the resulting optimum satisfies all the constraints, stop. Otherwise, set l=1 and go to Step 2.

Step 2. Activate any l constraints (i.e., convert them into equality) and optimize $f(x_t)$ subject to l active constraints by the Lagrangian method. If the resulting solution is feasible with respect to the remaining constraints, stop. It is a local optimum. Otherwise, activate another set of l constraints and repeat the step. If all sets of active constraints that have taken l at a time are considered without encountering a feasible solution, go to Step 3.

Step 3. If $l = s$ stop. No feasible solution exists. Otherwise, set $l = l+1$ and go to Step 2.

4.2.2. Joined entire price for Fuzzy Multi-item in Multi-outlet Distribution Network Inventory System

Throughout this study, the resulting decision variables and factors are profitable to lessen the effect of fuzzy quantities. Proceed $\tilde{D}_{gs}, \tilde{A}_{gs}, \tilde{S}_{gs}, \tilde{r}_{gs}, \tilde{P}_{gs}, \tilde{C}_{vgs}, \tilde{C}_{bgs}, V\tilde{E}C_{bgs}$, and $V\tilde{T}C_{bgs}$ are fuzzy factors. Presently, fuzzy multi-item in a multi-outlet distribution network inventory system brings together fuzzy order size $\tilde{Q}$ to be a hexagonal fuzzy number $\tilde{Q}_{gs} = (Q_{gs1}, Q_{gs2}, Q_{gs3}, Q_{gs4}, Q_{gs5}, Q_{gs6})$ with constraint $0 < Q_{gs1} \leq Q_{gs2} \leq Q_{gs3} \leq Q_{gs4} \leq Q_{gs5} \leq Q_{gs6}$.

The fuzzy total price of the multi-item in $s-$th outlet is

$$\begin{aligned} I\tilde{T}CMO_{gs}(\tilde{Q}_{gs},L_{gs},m_{gs}) &= (\tilde{D}_{gs}\oslash\tilde{Q}_{gs})\otimes(\tilde{A}_{gs}\oplus(\tilde{S}_{gs}\oslash m_{gs})\oplus R(L_{gs}))]\ominus[[(\tilde{Q}_{gs}\otimes\tilde{r}_{gs}\otimes\tilde{C}_{vgs})\oslash 2]\otimes[(m_{gs}\otimes\tilde{D}_{gs}\oslash\tilde{P}_{gs})+1] \\ &\oplus[[(\tilde{Q}_{gs}\otimes\tilde{r}_{gs})\oslash 2]\otimes[[m_{gs}\oplus((2\otimes\tilde{D}_{gs})\oslash\tilde{P}_{gs})]\otimes\tilde{C}_{vgs}\oplus\tilde{C}_{bgs}]]\oplus[\tilde{r}_{gs}\otimes\tilde{C}_{bgs}\otimes k_{gs}\otimes\sigma_{gs}\otimes\sqrt{L_{gs}}] \\ &+m_{gs}\otimes(FTC_{vgs}+V\tilde{T}C_{vgs})+(m_{gs}\otimes FEC_{vgs}+\tilde{Q}_{gs}\otimes V\tilde{E}C_{vgs}). \end{aligned} \tag{14}$$

Fuzzy operations for addition, subtraction, multiplication, and division ($\oplus$, $\ominus$, $\otimes$, and $\oslash$) are defined here using the function principle.

Assume $\tilde{D}=(D_{gs1}, D_{gs2}, D_{gs3}, D_{gs4}, D_{gs5}, D_{gs6})$, $\tilde{A}_{gs}=(A_{gs1}, A_{gs2}, A_{gs3}, A_{gs4}, A_{gs5}, A_{gs6})$, $\tilde{r}_{gs}=(r_{gs1}, r_{gs2}, r_{gs3}, r_{gs4}, r_{gs5}, r_{gs6})$, $\tilde{S}_{gs}=(S_{gs1}, S_{gs2}, S_{gs3}, S_{gs4}, S_{gs5}, S_{gs6})$, $\tilde{P}_{gs}=(P_{gs1}, P_{gs2}, P_{gs3}, P_{gs4}, P_{gs5}, P_{gs6})$, $\tilde{C}_{vgs}=(C_{vgs1}, C_{vgs2}, C_{vgs3}, C_{vgs4}, C_{vgs5}, C_{vgs6})$, $\tilde{C}_{bgs}=(C_{bgs1}, C_{bgs2}, C_{bgs3}, C_{bgs4}, C_{bgs5}, C_{bgs6})$, $V\tilde{E}C_{bgs}=(VEC_{bgs1}, VEC_{bgs2}, VEC_{bgs3}, VEC_{bgs4}, VEC_{bgs5}, VEC_{bgs6})$, and $V\tilde{T}C_{bgs} = (VTC_{bgs1}, VTC_{bgs2}, VTC_{bgs3}, VTC_{bgs4}, VTC_{bgs5}, VTC_{bgs6})$ are positive hexagonal fuzzy quantities. Based on equation (14), we determine the optimal quantity for every item and outlet in the following manner.

Fuzzy total price for each item of every outlet $I\tilde{T}CMO_{gs}(\tilde{Q}_{gs},L_{gs},m_{gs})$ is given in equation (14). Then,

$$\begin{aligned} I\tilde{T}CMO_{gs}(\tilde{Q}_{gs},L_{gs},m_{gs}) = \Bigg(&\frac{D_{gs1}}{Q_{gs6}}\left(A_{gs1}+\frac{S_{gs1}}{m_{gs}}+R(L_{gs})\right)-\frac{Q_{gs6}r_{gs6}C_{vgs6}}{2}\left(\frac{m_{gs}D_{gs6}}{P_{gs1}}+1\right)+\frac{Q_{gs1}r_{gs1}}{2}\left(\left(m_{gs}+\frac{2D_{gs1}}{P_{gs6}}\right)C_{vgs1}+C_{bgs1}\right) \\ &+r_{gs1}C_{bgs1}k_{gs}\sigma_{gs}\sqrt{L_{gs}}+Q_{gs1}VEC_{vgs1}+m_{gs}(FTC_{vgs}+VTC_{vgs1}+FEC_{vgs})\Bigg), \\ \Bigg(&\frac{D_{gs2}}{Q_{gs5}}\left(A_{gs2}+\frac{S_{gs2}}{m_{gs}}+R(L_{gs})\right)-\frac{Q_{gs5}r_{gs5}C_{vgs5}}{2}\left(\frac{m_{gs}D_{gs5}}{P_{gs2}}+1\right)+\frac{Q_{gs2}r_{gs2}}{2}\left(\left(m_{gs}+\frac{2D_{gs2}}{P_{gs5}}\right)C_{vgs2}+C_{bgs2}\right) \\ &+r_{gs2}C_{bgs2}k_{gs}\sigma_{gs}\sqrt{L_{gs}}+Q_{gs2}VEC_{vgs2}+m_{gs}(FTC_{vgs}+VTC_{vgs2}+FEC_{vgs})\Bigg), \\ \Bigg(&\frac{D_{gs3}}{Q_{gs4}}\left(A_{gs3}+\frac{S_{gs3}}{m_{gs}}+R(L_{gs})\right)-\frac{Q_{gs4}r_{gs4}C_{vgs4}}{2}\left(\frac{m_{gs}D_{gs4}}{P_{gs3}}+1\right)+\frac{Q_{gs3}r_{gs3}}{2}\left(\left(m_{gs}+\frac{2D_{gs3}}{P_{gs4}}\right)C_{vgs3}+C_{bgs3}\right) \\ &+r_{gs3}C_{bgs3}k_{gs}\sigma_{gs}\sqrt{L_{gs}}+Q_{gs3}VEC_{vgs3}+m_{gs}(FTC_{vgs}+VTC_{vgs3}+FEC_{vgs})\Bigg), \\ \Bigg(&\frac{D_{gs4}}{Q_{gs3}}\left(A_{gs4}+\frac{S_{gs4}}{m_{gs}}+R(L_{gs})\right)-\frac{Q_{gs3}r_{gs3}C_{vgs3}}{2}\left(\frac{m_{gs}D_{gs3}}{P_{gs4}}+1\right)+\frac{Q_{gs4}r_{gs4}}{2}\left(\left(m_{gs}+\frac{2D_{gs4}}{P_{gs3}}\right)C_{vgs4}+C_{bgs4}\right) \\ &+r_{gs4}C_{bgs4}k_{gs}\sigma_{gs}\sqrt{L_{gs}}+Q_{gs4}VEC_{vgs4}+m_{gs}(FTC_{vgs}+VTC_{vgs4}+FEC_{vgs})\Bigg), \\ \Bigg(&\frac{D_{gs5}}{Q_{gs2}}\left(A_{gs5}+\frac{S_{gs5}}{m_{gs}}+R(L_{gs})\right)-\frac{Q_{gs2}r_{gs2}C_{vgs2}}{2}\left(\frac{m_{gs}D_{gs2}}{P_{gs5}}+1\right)+\frac{Q_{gs5}r_{gs5}}{2}\left(\left(m_{gs}+\frac{2D_{gs5}}{P_{gs2}}\right)C_{vgs5}+C_{bgs5}\right) \\ &+r_{gs5}C_{bgs5}k_{gs}\sigma_{gs}\sqrt{L_{gs}}+Q_{gs5}VEC_{vgs5}+m_{gs}(FTC_{vgs}+VTC_{vgs5}+FEC_{vgs})\Bigg), \\ \Bigg(&\frac{D_{gs6}}{Q_{gs1}}\left(A_{gs6}+\frac{S_{gs6}}{m_{gs}}+R(L_{gs})\right)-\frac{Q_{gs1}r_{gs1}C_{vgs1}}{2}\left(\frac{m_{gs}D_{gs1}}{P_{gs6}}+1\right)+\frac{Q_{gs6}r_{gs6}}{2}\left(\left(m_{gs}+\frac{2D_{gs6}}{P_{gs1}}\right)C_{vgs6}+C_{bgs6}\right) \\ &+r_{gs6}C_{bgs6}k_{gs}\sigma_{gs}\sqrt{L_{gs}}+Q_{gs6}VEC_{vgs6}+m_{gs}(FTC_{vgs}+VTC_{vgs6}+FEC_{vgs})\Bigg). \end{aligned} \tag{15}$$

Applying equation (13) allows for the determination of the $I\tilde{T}CMO_{gs}(\tilde{Q}_{gs}, L_{gs}, m_{gs})$ through the Alpha cut technique, expressed as:

$$P(I\tilde{T}CMO_{gs}(\tilde{Q}_{gs},L_{gs},m_{gs}))=\Bigg[\frac{1}{4}\Bigg(\frac{D_{gs1}}{Q_{gs6}}\bigg(A_{gs1}+\frac{S_{gs1}}{m_{gs}}+R(L_{gs})\bigg)-\frac{Q_{gs6}r_{gs6}C_{vgs6}}{2}\bigg(\frac{m_{gs}D_{gs6}}{P_{gs1}}+1\bigg)+\frac{Q_{gs1}r_{gs1}}{2}\bigg(\bigg(m_{gs}+\frac{2D_{gs1}}{P_{gs6}}\bigg)C_{vgs1}+C_{bgs1}$$
$$+r_{gs1}C_{bgs1}k_{gs}\sigma_{gs}\sqrt{L_{gs}}+Q_{gs1}VEC_{vgs1}+m_{gs}(FTC_{vgs}+VTC_{vgs1}+FEC_{vgs})\Bigg)$$
$$+\frac{1}{4}\Bigg(\frac{D_{gs2}}{Q_{gs5}}\bigg(A_{gs2}+\frac{S_{gs2}}{m_{gs}}+R(L_{gs})\bigg)-\frac{Q_{gs5}r_{gs5}C_{vgs5}}{2}\bigg(\frac{m_{gs}D_{gs5}}{P_{gs2}}+1\bigg)+\frac{Q_{gs2}r_{gs2}}{2}\bigg(\bigg(m_{gs}+\frac{2D_{gs2}}{P_{gs5}}\bigg)C_{vgs2}+C_{bgs2}$$
$$+r_{gs2}C_{bgs2}k_{gs}\sigma_{gs}\sqrt{L_{gs}}+Q_{gs2}VEC_{vgs2}+m_{gs}(FTC_{vgs}+VTC_{vgs2}+FEC_{vgs})\Bigg)$$
$$+\frac{1}{4}\Bigg(\frac{D_{gs5}}{Q_{gs2}}\bigg(A_{gs5}+\frac{S_{gs5}}{m_{gs}}+R(L_{gs})\bigg)-\frac{Q_{gs2}r_{gs2}C_{vgs2}}{2}\bigg(\frac{m_{gs}D_{gs2}}{P_{gs5}}+1\bigg)+\frac{Q_{gs5}r_{gs5}}{2}\bigg(\bigg(m_{gs}+\frac{2D_{gs5}}{P_{gs2}}\bigg)C_{vgs5}+C_{bgs5}$$
$$+r_{gs5}C_{bgs5}k_{gs}\sigma_{gs}\sqrt{L_{gs}}+Q_{gs5}VEC_{vgs5}+m_{gs}(FTC_{vgs}+VTC_{vgs5}+FEC_{vgs})\Bigg)$$
$$+\frac{1}{4}\Bigg(\frac{D_{gs6}}{Q_{gs1}}\bigg(A_{gs6}+\frac{S_{gs6}}{m_{gs}}+R(L_{gs})\bigg)-\frac{Q_{gs1}r_{gs1}C_{vgs1}}{2}\bigg(\frac{m_{gs}D_{gs1}}{P_{gs6}}+1\bigg)+\frac{Q_{gs6}r_{gs6}}{2}\bigg(\bigg(m_{gs}+\frac{2D_{gs6}}{P_{gs1}}\bigg)C_{vgs6}+C_{bgs6}$$
$$+r_{gs6}C_{bgs6}k_{gs}\sigma_{gs}\sqrt{L_{gs}}+Q_{gs6}VEC_{vgs6}+m_{gs}(FTC_{vgs}+VTC_{vgs6}+FEC_{vgs})\Bigg)\Bigg]. \quad (16)$$

The joined entire price of $s-$th outlet for all G items is denoted by $I\tilde{T}CMO_s$, that is $I\tilde{T}CMO_s = \sum_{g=1}^{t_s} P(I\tilde{T}CMO_{gs}(\tilde{Q}_{gs}, L_{gs}, m_{gs}))$ for *s = 1,2,....,0* and the fuzzy joined entire price of the entire system is

$$I\tilde{T}CMO = \sum_{s=1}^{O} I\tilde{T}CMO_s = \sum_{s=1}^{O}\sum_{g=1}^{t_s} P(I\tilde{T}CMO_{gs}(\tilde{Q}_{gs}, L_{gs}, m_{gs})). \quad (17)$$

with $0 < Q_{gs1} \le Q_{gs2} \le Q_{gs5} \le Q_{gs6}$. Exchange the inequality condition $0 < Q_{gs1} \le Q_{gs2} \le Q_{gs5} \le Q_{gs6}$ with $Q_{gs2} - Q_{gs1} \ge 0,\ Q_{gs5} - Q_{gs2} \ge 0,\ Q_{gs6} - Q_{gs5} \ge 0,$ and $Q_{gs1} > 0,$ equation (16) remain valid and unchanged for this scenario.

The optimization of equation (16) is achieved by applying Lagrangian multipliers referencing the technique by Taha [17] to determine the values of $Q_{gs1}, Q_{gs2}, Q_{gs5}$, and Q_{gs6} necessary for minimizing the $P(I\tilde{T}CMO_{gs}(\tilde{Q}_{gs}, L_{gs}, m_{gs}))$.

Step 1. Optimizing the unconstrained system: To determine the minimum of $P(I\tilde{T}CMO_{gs}(\tilde{Q}_{gs}, L_{gs}, m_{gs}))$, the partial derivatives of $P(I\tilde{T}CMO_{gs}(\tilde{Q}_{gs}, L_{gs}, m_{gs}))$ are found with respect to $Q_{gs1}, Q_{gs2}, Q_{gs5}$, and Q_{gs6} then equate them to zero as follows.

$$\frac{1}{4}\left[-\frac{D_{gs6}}{Q_{gs1}{}^{2}}\left(A_{gs6}+\frac{S_{gs6}}{m_{gs}}+R(L_{gs})\right)-\frac{r_{gs1}C_{vgs1}}{2}\left(\frac{m_{gs}D_{gs1}}{P_{gs6}}+1\right)+\frac{r_{gs1}}{2}\left(\left(m_{gs}+\frac{2D_{gs1}}{P_{gs6}}\right)C_{vgs1}+C_{bgs1}\right)+VEC_{vgs1}\right]=0 \tag{18}$$

$$\frac{1}{4}\left[-\frac{D_{gs5}}{Q_{gs2}{}^{2}}\left(A_{gs5}+\frac{S_{gs5}}{m_{gs}}+R(L_{gs})\right)-\frac{r_{gs2}C_{vgs2}}{2}\left(\frac{m_{gs}D_{gs2}}{P_{gs5}}+1\right)+\frac{r_{gs2}}{2}\left(\left(m_{gs}+\frac{2D_{gs2}}{P_{gs5}}\right)C_{vgs2}+C_{bgs2}\right)+VEC_{vgs2}\right]=0 \tag{19}$$

$$\frac{1}{4}\left[-\frac{D_{gs2}}{Q_{gs5}{}^{2}}\left(A_{gs2}+\frac{S_{gs2}}{m_{gs}}+R(L_{gs})\right)-\frac{r_{gs5}C_{vgs5}}{2}\left(\frac{m_{gs}D_{gs5}}{P_{gs2}}+1\right)+\frac{r_{gs4}}{2}\left(\left(m_{gs}+\frac{2D_{gs5}}{P_{gs2}}\right)C_{vgs5}+C_{bgs5}\right)+VEC_{vgs5}\right]=0 \tag{20}$$

$$\frac{1}{4}\left[-\frac{D_{gs1}}{Q_{gs6}{}^{2}}\left(A_{gs1}+\frac{S_{gs1}}{m_{gs}}+R(L_{gs})\right)-\frac{r_{gs6}C_{vgs6}}{2}\left(\frac{m_{gs}D_{gs6}}{P_{gs1}}+1\right)+\frac{r_{gs6}}{2}\left(\left(m_{gs}+\frac{2D_{gs6}}{P_{gs1}}\right)C_{vgs6}+C_{bgs6}\right)+VEC_{vgs6}\right]=0 \tag{21}$$

By solving equations (18) to (21), the optimum order quantities Q_{gs1}, Q_{gs2}, Q_{gs5}, and Q_{gs6} are obtained and given as

$$Q_{gs1}=\sqrt{\frac{2D_{gs6}\left(A_{gs6}+\frac{S_{gs6}}{m_{gs}}+R(L_{gs})\right)}{r_{gs1}\left(\left(m_{gs}\left(1-\frac{D_{gs1}}{P_{gs6}}\right)-1+\frac{2D_{gs1}}{P_{gs6}}\right)C_{vgs1}+C_{bgs1}\right)+2VEC_{vgs1}}}, \tag{22}$$

$$Q_{gs2}=\sqrt{\frac{2D_{gs5}\left(A_{gs5}+\frac{S_{gs5}}{m_{gs}}+R(L_{gs})\right)}{\left[r_{gs2}\left(\left(m_{gs}\left(1-\frac{D_{gs2}}{P_{gs5}}\right)-1+\frac{2D_{gs2}}{P_{gs5}}\right)C_{vgs2}+C_{bgs2}\right)+2VEC_{vgs2}\right]}}, \tag{23}$$

$$Q_{gs5}=\sqrt{\frac{2D_{gs2}\left(A_{gs2}+\frac{S_{gs2}}{m_{gs}}+R(L_{gs})\right)}{\left[r_{gs5}\left(\left(m_{gs}\left(1-\frac{D_{gs5}}{P_{gs2}}\right)-1+\frac{2D_{gs5}}{P_{gs2}}\right)C_{vgs5}+C_{bgs5}\right)+2VEC_{vgs5}\right]}}, \tag{24}$$

$$Q_{gs6}=\sqrt{\frac{2D_{gs1}\left(A_{gs1}+\frac{S_{gs1}}{m_{gs}}+R(L_{gs})\right)}{\left[r_{gs6}\left(\left(m_{gs}\left(1-\frac{D_{gs6}}{P_{gs1}}\right)-1+\frac{2D_{gs6}}{P_{gs1}}\right)C_{vgs6}+C_{bgs6}\right)+2VEC_{vgs6}\right]}}. \tag{25}$$

The above results show that $Q_{gs1}>Q_{gs2}>Q_{gs5}>Q_{gs6}$ and do not satisfy the constraint $Q_{gs6}\geq Q_{gs5}\geq Q_{gs2}\geq Q_{gs1}>0$ Therefore, set $k=1$ and go to Step 2.

Step 2. Convert the inequality constraint $Q_{gs2}-Q_{gs1}\geq 0$ into equality constrain $Q_{gs2}-Q_{gs1}=0$ and optimize $P[I\tilde{T}\ CMO_{gs}(\tilde{Q}_{gs}, L_{gs}, m_{gs})]$ subject to $Q_{gs2}-Q_{gs1}=0$

through Lagrangian Process.

The Lagrangian function is $L(Q_{gs1}, Q_{gs2}, Q_{gs5}, Q_{gs6}, \lambda_{gs})=P[I\tilde{T}CMO_{gs}(\tilde{Q}_{gs}, L_{gs}, m_{gs})] - \lambda_{gs}(Q_{gs2} - Q_{gs1})$.

The partial derivatives of $L(Q_{gs1}, Q_{gs2}, Q_{gs5}, Q_{gs6}, \lambda_{gs})$with respect to Q_{gs1}, Q_{gs2}, Q_{gs5}, Q_{gs6} and λ_{gs} are engaged to acquire the minimum of $L(Q_{gs1}, Q_{gs2}, Q_{gs5}, Q_{gs6}, \lambda_{gs})$

Let all the partial derivatives be equal to zero to solve Q_{gs1}, Q_{gs2}, Q_{gs5} and Q_{gs6}. Then,

$$Q_{gs1}=Q_{gs2}=\sqrt{\frac{2D_{gs6}\left(A_{gs6}+\frac{S_{gs6}}{m_{gs}}+R(L_{gs})\right)+2D_{gs5}\left(A_{gs5}+\frac{S_{gs5}}{m_{gs}}+R(L_{gs})\right)}{\left[r_{gs1}\left(\left(m_{gs}\left(1-\frac{D_{gs1}}{P_{gs6}}\right)-1+\frac{2D_{gs1}}{P_{gs6}}\right)C_{vgs1}+C_{bgs1}\right)+2VEC_{vgs1}\right]+\left[r_{gs2}\left(\left(m_{gs}\left(1-\frac{D_{gs2}}{P_{gs5}}\right)-1+\frac{2D_{gs2}}{P_{gs5}}\right)C_{vgs2}+C_{bgs2}\right)+2VEC_{vgs2}\right]}}, \tag{26}$$

$$Q_{gs5}=\sqrt{\frac{2D_{gs2}\left(A_{gs2}+\frac{S_{gs2}}{m_{gs}}+R(L_{gs})\right)}{\left[r_{gs5}\left(\left(m_{gs}\left(1-\frac{D_{gs5}}{P_{gs2}}\right)-1+\frac{2D_{gs5}}{P_{gs2}}\right)C_{vgs5}+C_{bgs5}\right)+2VEC_{vgs5}\right]}}, \tag{27}$$

$$Q_{gs6}=\sqrt{\frac{2D_{gs1}\left(A_{gs1}+\frac{S_{gs1}}{m_{gs}}+R(L_{gs})\right)}{\left[r_{gs6}\left(\left(m_{gs}\left(1-\frac{D_{gs6}}{P_{gs1}}\right)-1+\frac{2D_{gs6}}{P_{gs1}}\right)C_{vgs6}+C_{bgs6}\right)+2VEC_{vgs6}\right]}}, \tag{28}$$

Again, the above results show that $Q_{gs6}<Q_{gs5}$ and it does not satisfy the constraint $Q_{gs6}\geq Q_{gs5}\geq Q_{gs2}\geq Q_{gs1}>0$. Consequently, the current solution does not represent a local optimum. This conclusion holds regardless of which inequality constraint is treated as an equality. By applying this logic to the remaining constraints, we set $k = 2$ and proceed to Step 3.

Step 3. To evaluate the conditions, the non-negativity constraints $Q_{gs2}—Q_{gs1}\geq 0$, and $Q_{gs5}—Q_{gs2}\geq 0$ are treated as active equality constraints, such that $Q_{gs2}—Q_{gs1}=0$ and $Q_{gs5}—Q_{gs2}=0$. $P[I\tilde{T}\ CMO_{gs}(\tilde{Q}_{gs}, L_{gs}, m_{gs})]$ is optimized subject to $Q_{gs2}—Q_{gs1}=0$ and $Q_{gs5}—Q_{gs2}=0$ by the Lagrangian process.

Then the Lagrangian function is $L(Q_{gs1}, Q_{gs2}, Q_{gs5}, Q_{gs6}, \lambda_{gs1}, \lambda_{gs2})=P[I\tilde{T}\ CMO_{gs}(\tilde{Q}_{gs}, L_{gs}, m_{gs})] - \lambda_{gs1}(Q_{gs2} - Q_{gs1}) - \lambda_{gs2}(Q_{gs5} - Q_{gs2})$.

To determine the minimum value of the Lagrangian function $L(Q_{gs1}, Q_{gs2}, Q_{gs5}, Q_{gs6}, \lambda_{gs1}, \lambda_{gs2})$, the partial derivatives of $L(Q_{gs1}, Q_{gs2}, Q_{gs5}, Q_{gs6}, \lambda_{gs1}, \lambda_{gs2})$ are taken with respect to $Q_{gs1}, Q_{gs2}, Q_{gs5}, Q_{gs6}, \lambda_{gs1}, \lambda_{gs2}$ and equating the partial derivatives to zero allows us to solve the resulting system for the decision variables $Q_{gs1}, Q_{gs2}, Q_{gs5}$ and Q_{gs6}, then we get,

$$Q_{gs1}=Q_{gs2}=Q_{gs5}=\sqrt{\frac{2D_{gs6}\left(A_{gs6}+\frac{S_{gs6}}{m_{gs}}+R(L_{gs})\right)+2D_{gs5}\left(A_{gs5}+\frac{S_{gs5}}{m_{gs}}+R(L_{gs})\right)+2D_{gs2}\left(A_{gs2}+\frac{S_{gs2}}{m_{gs}}+R(L_{gs})\right)}{\begin{array}{l}\left[r_{g1}\left(\left(m_{gs}\left(1-\frac{D_{gs1}}{P_{gs6}}\right)-1+\frac{2D_{gs1}}{P_{gs6}}\right)C_{vgs1}+C_{bgs1}\right)+2VEC_{vgs1}\right]+\left[r_{g2}\left(\left(m_{gs}\left(1-\frac{D_{gs2}}{P_{gs5}}\right)-1+\frac{2D_{gs2}}{P_{gs5}}\right)C_{vgs2}+C_{bgs2}\right)+2VEC_{vgs2}\right]\\+\left[r_{gs5}\left(\left(m_{gs}\left(1-\frac{D_{gs5}}{P_{gs2}}\right)-1+\frac{2D_{gs5}}{P_{gs2}}\right)C_{vgs5}+C_{bgs5}\right)+2VEC_{vgs5}\right]\end{array}}}, \tag{29}$$

$$Q_{gs6}=\sqrt{\frac{2D_{gs1}\left(A_{gs1}+\frac{S_{gs1}}{m_{gs}}+R(L_{gs})\right)}{\left[r_{gs6}\left(\left(m_{gs}\left(1-\frac{D_{gs6}}{P_{gs1}}\right)-1+\frac{2D_{gs6}}{P_{gs1}}\right)C_{vgs6}+C_{bgs6}\right)+2VEC_{vgs6}\right]}}. \tag{30}$$

The above results show that $Q_{gs1}>Q_{gs6}$ does not satisfy the constraint $Q_{gs6}\geq Q_{gs5}\geq Q_{gs2}\geq Q_{gs1}>0$. Consequently, the current solution fails to meet the criteria for a local optimum. This outcome remains consistent even when any two inequality constraints are treated as equalities. Moving forward, we set $k=3$ and proceed to Step 4.

Step 4. Change the inequality constraints $Q_{gs2}-Q_{gs1}\geq 0, Q_{gs5}-Q_{gs2}\geq 0$ and $Q_{gs6}-Q_{gs5}\geq 0$ into equality constraints $Q_{gs2}-Q_{gs1}=0, Q_{gs5}-Q_{gs2}=0$ and $Q_{gs6}-Q_{gs5}=0$ $P[I\tilde{T}CMO_{gs}(\tilde{Q}_{gs}, L_{gs}, m_{gs})]$ is optimized subject to $Q_{gs2}-Q_{gs1}=0$ $Q_{gs5}-Q_{gs2}=0$and $Q_{gs6}-Q_{gs5}=0$ by the Lagrangian Process.

The Lagrangian function is given as $L(Q_{gs1}, Q_{gs2}, Q_{gs5}, Q_{gs6}, \lambda_{gs1}, \lambda_{gs2}, \lambda_{gs3})=P[I\tilde{T}CMO_{gs}(\tilde{Q}_{gs}, L_{gs}, m_{gs})]-\lambda_{gs1}(Q_{gs2}-Q_{gs1})-\lambda_{gs2}(Q_{gs5}-Q_{gs2})-\lambda_{gs3}(Q_{gs6}-Q_{gs5})$

To optimize $L(Q_{gs1}, Q_{gs2}, Q_{gs5}, Q_{gs6}, \lambda_{gs1}, \lambda_{gs2}, \lambda_{gs3})$ for minimum cost, the partial derivatives of $L(Q_{gs1}, Q_{gs2}, Q_{gs5}, Q_{gs6}, \lambda_{gs1}, \lambda_{gs2}, \lambda_{gs3})$ are taken with respect to $Q_{gs1}, Q_{gs2}, Q_{gs5}, Q_{gs6}, \lambda_{gs1}, \lambda_{gs2}, \lambda_{gs3}$ and we determine the stationary points of the function by equating the partial derivatives with respect to $Q_{gs1}, Q_{gs2}, Q_{gs5}$ and Q_{gs6} to zero.

The result is,

$$Q_{gs1}=Q_{gs2}=Q_{gs5}=Q_{gs6}=\sqrt{\frac{2D_{gs6}\left(A_{gs6}+\frac{S_{gs6}}{m_{gs}}+R(L_{gs})\right)+2D_{gs5}\left(A_{gs5}+\frac{S_{gs5}}{m_{gs}}+R(L_{gs})\right)+2D_{gs2}\left(A_{gs2}+\frac{S_{gs2}}{m_{gs}}+R(L_{gs})\right)+2D_{gs1}\left(A_{gs1}+\frac{S_{gs1}}{m_{gs}}+R(L_{gs})\right)}{\left[r_{gs1}\left(\left(m_{gs}\left(1-\frac{D_{gs1}}{P_{gs6}}\right)-1+\frac{2D_{gs1}}{P_{gs6}}\right)C_{vgs1}+C_{bgs1}\right)+2VEC_{vgs1}\right]+\left[r_{gs2}\left(\left(m_{gs}\left(1-\frac{D_{gs2}}{P_{gs5}}\right)-1+\frac{2D_{gs2}}{P_{gs5}}\right)C_{vgs2}+C_{bgs2}\right)+2VEC_{vgs2}\right]+\left[r_{gs5}\left(\left(m_{gs}\left(1-\frac{D_{gs5}}{P_{gs2}}\right)-1+\frac{2D_{gs5}}{P_{gs2}}\right)C_{vgs5}+C_{bgs5}\right)+2VEC_{vgs5}\right]+\left[r_{gs6}\left(\left(m_{gs}\left(1-\frac{D_{gs6}}{P_{gs1}}\right)-1+\frac{2D_{gs6}}{P_{gs1}}\right)C_{vgs6}+C_{bgs6}\right)+2VEC_{vgs6}\right]}}. \quad (31)$$

The procedure terminates here, as the values for $\tilde{Q}_{gs}=(Q_{gs1}, Q_{gs2}, Q_{gs5}, Q_{gs6})$ satisfy every boundary condition, establishing a local optimum. This result is identified as the unique feasible solution to the model's objective function. Consequently, it serves as the global optimum for the fuzzy-parameter system under the Lagrangian framework. By equating the order quantities $\tilde{Q}^*_{gs}=Q_{gs1}=Q_{gs2}=Q_{gs5}=Q_{gs6}$.The optimum fuzzy order size is $\tilde{Q}^*_{gs}=(Q^*_{gs}, Q^*_{gs}, Q^*_{gs}, Q^*_{gs})$ where

$$\tilde{Q}^*_{gs}=\sqrt{\frac{2D_{gs6}\left(A_{gs6}+\frac{S_{gs6}}{m_{gs}}+R(L_{gs})\right)+2D_{gs5}\left(A_{gs5}+\frac{S_{gs5}}{m_{gs}}+R(L_{gs})\right)+2D_{gs2}\left(A_{gs2}+\frac{S_{gs2}}{m_{gs}}+R(L_{gs})\right)+2D_{gs1}\left(A_{gs1}+\frac{S_{gs1}}{m_{gs}}+R(L_{gs})\right)}{\left[r_{gs1}\left(\left(m_{gs}\left(1-\frac{D_{gs1}}{P_{gs6}}\right)-1+\frac{2D_{gs1}}{P_{gs6}}\right)C_{vgs1}+C_{bgs1}\right)+2VEC_{vgs1}\right]+\left[r_{gs2}\left(\left(m_{gs}\left(1-\frac{D_{gs2}}{P_{gs5}}\right)-1+\frac{2D_{gs2}}{P_{gs5}}\right)C_{vgs2}+C_{bgs2}\right)+2VEC_{vgs2}\right]+\left[r_{gs5}\left(\left(m_{gs}\left(1-\frac{D_{gs5}}{P_{gs2}}\right)-1+\frac{2D_{gs5}}{P_{gs2}}\right)C_{vgs5}+C_{bgs5}\right)+2VEC_{vgs5}\right]+\left[r_{gs6}\left(\left(m_{gs}\left(1-\frac{D_{gs6}}{P_{gs1}}\right)-1+\frac{2D_{gs6}}{P_{gs1}}\right)C_{vgs6}+C_{bgs6}\right)+2VEC_{vgs6}\right]}}. \quad (32)$$

4.2.3. Optimization Algorithm

The succeeding optimization algorithm is considered to obtain the crisp and fuzzy optimum order size for each item for the specific outlet. Then, the minimum joined entire price for the entire system is found.

Algorithm 1

Step 1. Use differential calculus optimization method, find Q^*_{gs} then calculate Q^*_{gs} by equation (12).

Step 2. Substitute equation (12) into equation (8) yields the specific solution set $ITCMO_{gs}$ for the $g-$th item at the $s-$th outlet within the crisp joined inventory framework.

Step 3. Calculate $ITCMO_{gs}$, $ITCMO$ from equations (8) and (9) consistently.

Step 4. Use $(\alpha)-$cut and Extension of Lagrangian Process, find $\tilde{Q}^*_{gs}$ then calculate $\tilde{Q}^*_{gs}$ by equation (32).

Step 5. Substitute equation (32) into equation (16) yields the specific solution set $P(I\tilde{T}CMO_{gs})$ for the $g-$th item at the $s-$th outlet within the fuzzy joined inventory framework.

Step 6. Compute $P(I\tilde{T}CMO_{gs})I\tilde{T}CMO$ from equations (16) and (17) correspondingly.

Step 7. Repeat Steps 1 to 6 for each lead time $L_{gs}=3,4,6,8$.

Step 8. Evaluates the performance of the multi-outlet network by comparing the total system cost against the optimal order size of $g-$th item of $s-$th outlet. If the fuzzy parameters yield higher order quantities and integrated costs $Q^*_{gs}>\tilde{Q}^*_{gs}$ and $ITCMO_{gs}> P(I\tilde{T}CMO_{gs})$, the fuzzy model is designated as the optimal solution. Conversely, if the crisp variables result in lower values $Q^*_{gs}<\tilde{Q}^*_{gs}$ and $ITCMO_{gs}< P(I\tilde{T}CMO_{gs})$ the crisp model is prioritized for its precision.

Step 9. Examine the optimal order quantity for $g-$ th item of $s-$ th outlet specific items across various locations within a multi-item, multi-outlet framework. We contrast the total cost and the resulting economic savings derived from both the crisp and fuzzy mathematical models.

5. Numerical examples

Numerical circumstances are quantified to conclude the result procedure by employing the recommended algorithm. In this regard, the ideal multi-item in multi-outlet distribution network inventory system is established. The outcomes attained over MATLAB software. Then projected fuzzy multi-item in multi-outlet distribution network inventory system can be used in companies. The proposed integrated multi-item in multi-outlet distribution network inventory system is an additional operative designed at the supply chain companies' development of seller-outlet organisation.

Example. 1

Crisp Multi-item in Multi-outlet Distribution Network Inventory System

The outcomes establish crisp model with primary inputs from [12]. The residual inputs are allowed to the system.

Presume the seller starts business with showrooms in important places from where seller runs two outlets for selling furniture (cot, dining table, sofa, …) and electronic items (washing machine, fridge, television, …). Let us consider the joined multi-item in multi-outlet distribution network inventory system for five items and two outlets. From the first outlet, two types of furniture (wardrobe, dressing table) are retailed and three type of electronic items (refrigerator, laptop, air conditioner) are retailed from second outlet. That is items G=5, outlet 0=2. The parameters are FEC_{vgs}=\$0.2/shipment, FTC_{vgs}= \$0.2/shipment in place of whole s=1, 2 also g=1, 2, 3, 4, 5.

Table 1: Common outlet crisp input for all items

Outlet s	Item g	P_{gs}	C_{vgs}	C_{bgs}	r_{gs}	S_{gs}	VTC_{vgs}	VEC_{vgs}	A_{gs}	σ_{gs}	k_{gs}
1	1	3200	200	250	0.2	400	0.5	0.1	21.87	7	2.33
	2	4000	250	312.5	0.25	500	0.625	0.125	27.3375	8	2.097
2	1	4800	300	375	0.3	600	0.75	0.15	32.805	9	1.864
	2	5600	350	437.5	0.35	700	0.875	0.175	38.2725	10	1.631
	3	6400	400	500	0.4	800	1.0	0.2	43.74	11	1.398

Table 2: Demand for the items in outlets

D_{gs}	Outlet - 1		Outlet - 2		
Items	1000	1250	1500	1750	2000

Table 3: Common items data for all outlets

	L_{gs}	$R(L)_{gs}$	m_{gs}
Outlets	3	53.2	3
	4	18.2	4
	6	1.4	5
	8	0	5

Table 4: Crisp output of Example 1

Q_{11}	Q_{12}	Q_{21}	Q_{22}	Q_{23}	$ITCMO_1$	$ITCMO_2$	$ITCMO_1$+ $ITCMO_2$
59.6	58.0	57.0	56.2	55.6	21352.0	74395.0	95747.0
44.0	43.4	43.0	42.7	42.5	20471.0	72604.0	93075.0
34.6	34.6	34.5	34.5	34.5	20508.0	73732.0	94240.0
34.4	34.4	34.4	34.4	34.4	21222.0	76346.0	97568.0

In Table 1, certain data are specified and it is identical for entire outlets. Table 2 comprises demand of every item for all outlets and Table 3 comprises the lead time input for all outlets. Using algorithm 1, the crisp system's optimum outcomes are organised in Table 4.

Example. 2

Fuzzy Multi-item in Multi-outlet Distribution Network Inventory System

The inputs are same as in Example 1, except the fuzzy inputs that are given in Tables 5 and 6.

Table 5: Common outlet fuzzy input for all items

Fuzzy parameters	Fuzzy data
$\tilde{P}_{gs}$ (If s=1 then g=1, 2)	P_{gs1} = (2400, 3000), P_{gs2} = (2720, 3400), P_{gs3} = (3040, 3800), P_{gs4} = (3360, 4200), P_{gs5} = (3680, 4600), P_{gs6} = (4000, 5000).
$\tilde{P}_{gs}$ (If s=2 then g=1, 2, 3)	P_{gs1} = (3600, 4200, 4800), P_{gs2} = (4080, 4760, 5440), P_{gs3} = (4560, 5320, 6080), P_{gs4} = (5040, 5880, 6720), P_{gs5} = (5520, 6440, 7360), P_{gs6} = (6000, 7000, 8000).
$\tilde{C}_{vgs}$ (If s=1 then g=1, 2)	C_{vgs1} = (150, 187.5), C_{vgs2} =(170, 212.5), C_{vgs3} =(190, 237.5), C_{vgs4} =(210, 262.5), C_{vgs5} =(230, 287.5), C_{vgs6} =(250, 312.5).
$\tilde{C}_{vgs}$ (If s=2 then g=1, 2, 3)	C_{vgs1} = (225, 262.5, 300), C_{vgs2} = (255, 297.5, 340), C_{vgs3} = (285, 332.5, 380), C_{vgs4} = (315, 367.5, 420), C_{vgs5} = (345, 402.5, 460), C_{vgs6} (375, 437.5, 500).
$\tilde{C}_{bgs}$ (If s=1 then g=1, 2)	C_{bgs1}=(187.5, 234.375), C_{bgs2}=(212.5, 265.63), C_{bgs3}=(237.5, 296.88), C_{bgs4}=(262.5, 328.13), C_{bgs5}=(287.5, 359.38), C_{bgs6}=(312.5, 390.625).
$\tilde{C}_{bgs}$ (If s=2 then g=1, 2, 3)	C_{bgs1}=(281.25, 328.125, 375), C_{bgs2}=(318.75, 371.88, 425), C_{bgs3}=(356.25, 415.63, 475), C_{bgs4}=(393.75, 459.38, 525), C_{bgs5}=(431.25, 503.13, 575), C_{bgs6}=(468.75, 546.875, 625).
$\tilde{r}_{gs}$ (If s=1 then g=1, 2)	r_{gs1}=(0.15,0.1875),r_{gs2}=(0.17, 0.2125), r_{gs3}=(0.19, 0.2375), r_{gs4}=(0.21, 0.2625), r_{gs5}=(0.23, 0.2875), r_{gs6}=(0.25, 0.3125).
$\tilde{r}_{gs}$ (If s=2 then g=1, 2, 3)	r_{gs1}=(0.225, 0.2625, 0.3), r_{gs2}=(0.255, 0.2975, 0.34), r_{gs3}=(0.285, 0.3325, 0.38), r_{gs4}=(0.315, 0.3675, 0.42), r_{gs5}=(0.345, 0.4025, 0.46), r_{gs6}=(0.375, 0.4375, 0.5).
$\tilde{S}_{gs}$ (If s=1 then g=1, 2)	S_{gs1}=(300, 375), S_{gs2}=(340, 425), S_{gs3}=(380, 475), S_{gs4}=(420, 525), S_{gs5}=(460, 575), S_{gs6}=(500, 625).
$\tilde{S}_{gs}$ (If s=2 then g=1, 2, 3)	S_{gs1}=(450, 525, 600), S_{gs2}=(510, 595, 680), S_{gs3}=(570, 665, 760), S_{gs4}=(630, 735, 840), S_{gs5}=(690, 805, 920), S_{gs6}=(750, 875, 1000).
$V\tilde{T}C_{vgs}$ (If s=1 then g=1, 2)	VTC_{vgs1}=(0.375, 0.46875), VTC_{vgs2}=(0.425, 0.5313), VTC_{vgs3}=(0.475, 0.5938), VTC_{vgs4}=(0.525, 0.6563), VTC_{vgs5}=(0.575, 0.7188), VTC_{vgs6}=(0.625, 0.78125).

Fuzzy parameters	Fuzzy data
$V\tilde{T}C_{vgs}$ (If s=2 then g=1, 2, 3)	VTC_{vgs1}=(0.5625, 0.65625, 0.75), VTC_{vgs2}=(0.6375, 0.7438, 0.85), VTC_{vgs3}=(0.7125, 0.8313, 0.95), VTC_{vgs4}=(0.7875, 0.9188, 1.05), VTC_{vgs5}=(0.8625, 1.0063, 1.15), VTC_{vgs6}=(0.9375, 1.09375, 1.25).
$V\tilde{E}C_{vgs}$ (If s=1 then g=1, 2)	VEC_{vgs1}=(0.075, 0.09375), VEC_{vgs2}=(0.085, 0.1063), VEC_{vgs3}=(0.095, 0.1188), VEC_{vgs4}=(0.105, 0.1313), VEC_{vgs5}=(0.115, 0.1438), VEC_{vgs6}=(0.125, 0.15625).
$V\tilde{E}C_{vgs}$ (If s=2 then g=1, 2, 3)	VEC_{vgs1}=(0.1125, 0.13125, 0.15), VEC_{vgs2}=(0.1275, 0.1488, 0.17), VEC_{vgs3}=(0.1425, 0.1663, 0.19), VEC_{vgs4}=(0.1575, 0.1838, 0.21), VEC_{vgs5}=(0.1725, 0.2013, 0.23), VEC_{vgs6}=(0.1875, 0.21875, 0.25).
$\tilde{A}_{gs}$ (If s=1 then g=1, 2)	A_{gs1}=(16.4025, 20.5031), A_{gs2}=(18.59, 23.237), A_{gs3}= (20.777, 25.971), A_{gs4}=(22.964, 28.704), A_{gs5}=(25.151, 31.438), A_{gs6}=(27.3375, 34.17188).
$\tilde{A}_{gs}$ (If s=2 then g=1, 2, 3)	A_{gs1}=(24.6038, 28.7044, 32.805), A_{gs2}=(27.884, 32.532, 37.179), A_{gs3}= (31.165, 36.359, 41.553), A_{gs4}=(34.445, 40.186, 45.927), A_{gs5}=(37.726, 44.013, 50.301), A_{gs6}=(41.00625, 47.84063, 54.675).

Table 6: Demand for g-th item in s-th outlet

Outlet s	Demand $\tilde{D}_{gs}$
If s=1 then g=1, 2.	D_{gs1}=(750, 937.5), D_{gs2}=(850, 1062.5), D_{gs3}=(950, 1187.5), D_{gs4}=(1050, 1312.5), D_{gs5}=(1150, 1437.5), D_{gs6}=(1250, 1562.5).
If s=2 then g=1, 2, 3.	D_{gs1}=(1125, 1312.5, 1500), D_{gs2}=(1275, 1487.5, 1700), D_{gs3}=(1425, 1662.5, 1900), D_{gs4}=(1575, 1837.5, 2100), D_{gs5}=(1725, 2012.5, 2300), D_{gs6}=(1875, 2187.5, 2500).

Table 7: Fuzzy output of example 2

$\tilde{Q}_{11}$	$\tilde{Q}_{12}$	$\tilde{Q}_{21}$	$\tilde{Q}_{22}$	$\tilde{Q}_{23}$	$I\tilde{T}CMO_1$	$I\tilde{T}CMO_2$	$I\tilde{T}CMO_1$+ $I\tilde{T}CMO_2$
60.0	58.5	57.5	56.8	56.2	21120.0	73709.0	94829.0
44.8	44.3	43.9	43.6	43.4	20236.0	71815.0	92051.0
35.6	35.6	35.5	35.5	35.5	20301.0	72960.0	93261.0
35.4	35.4	35.4	35.4	35.4	21057.0	75711.0	96768.0

Table 8: Summary of Crisp and Fuzzy outlet solutions

L_{gs}	Crisp System		Fuzzy System		Savings %	
	$ITCMO_1$	$ITCMO_2$	$I\tilde{T}CMO_1$	$I\tilde{T}CMO_2$	Outlet 1	Outlet 2
3	21352	74395	21120	73709	1.1	0.9
4	20471	72604	20236	71815	1.1	1.1
6	20508	73732	20301	72960	1.0	1.0
8	21222	76346	21057	75711	0.8	0.8

Using algorithm 1, the fuzzy system's results are depicted in Table 7. Table 8 shows saving percentage for each outlet's joined entire price of fuzzy system. The comparative analysis between the crisp system and the fuzzy system demonstrates a clear economic advantage for the fuzzy-logic-based approach. As detailed in Table 8, the fuzzy model consistently yields lower joined entire prices for multi-outlets $I\tilde{T}CMO$ across all tested operational levels L_{gs}.

For outlet 1, the fuzzy system achieved its maximum efficiency gain of 1.1% at L_{gs}=3 and L_{gs}=4, reducing costs from \$21352 to \$21120 and \$20471 to \$20236, respectively. Similarly, outlet 2 saw a peak saving of 1.1% at L_{gs} =4, with costs dropping from \$72604 to \$71815. Across both outlets and all levels, the savings remained consistent within a range of 0.8% to 1.1%.

5.1. Graphical Representation

Each outlet's joined entire price for disparate values of lead time and demand interrelated in the crisp and fuzzy organisations, as exposed in the graphical depiction of Figure 1. Each outlet's joined entire price $ITCMO_s$ and $I\tilde{T}CMO_s$ varies while the lead time rises. The bar graph further validates these findings by illustrating the cost optimizations.

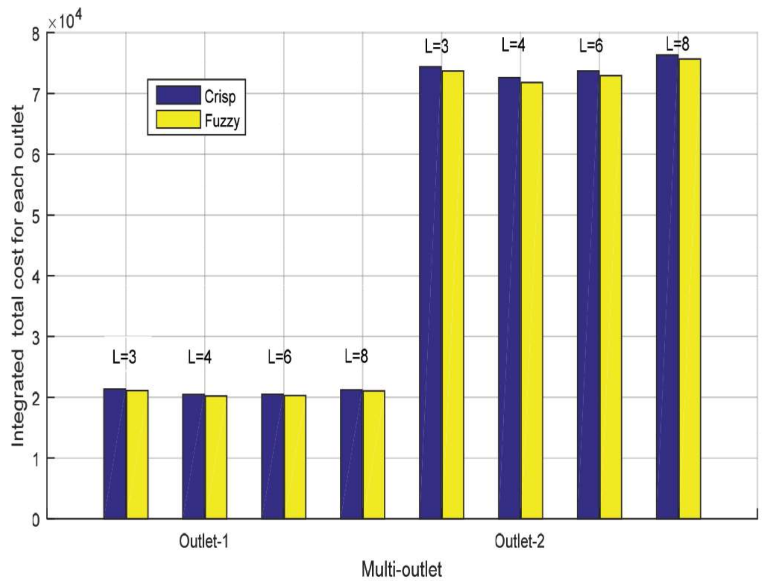

Figure 1: Integrated total cost for multi-item in multi-outlet versus lead time.

In every grouped bar for both outlet-1 and outlet-2, the fuzzy system are visibly shorter than the crisp system, providing immediate visual confirmation of the fuzzy system's superiority. While outlet-2 operates at a much higher cost \$76346 compared to outlet-1 \$21352, the fuzzy model adapts effectively to both scales, maintaining consistent percentage-based savings. The data suggests that by integrating fuzzy logic into the supply chain distribution network, the system can better account for variables that a crisp model treats as fixed. This results in a more optimized allocation of funds and a reliable reduction in total integrated costs. It is perceived that each outlet's joined entire price is professionally enriched in fuzzy system related to crisp system. The joined entire price for each outlet stays positively decreased in the fuzzy system with respect to crisp system.

6. Comparative study

Table 8 displays savings percentage of multi-item's optimum order size and each outlet's joined entire price for fuzzy system. In Table 9, the numerical results are presented. The optimum value of crisp multi-item in multi-outlet distribution network system's minimized joined entire price is \$380630. The optimum value of fuzzy multi-item in multi-outlet distribution network system's minimized joined entire price is \$376927. The comparative dissimilarities for crisp and fuzzy models in joined entire price of the systems can be grasped in Table 9. By comparing crisp and fuzzy multi–item in multi-outlet distribution network inventory model as well as joined entire price, the savings percentage is 0.97%.

Fuzzy system supports the trades to achieve uncertain inventory cost restrictions. Uncertain cost constraints of inventory administration systems are established to be hopeful and marginally weighty. In this regard, admins are proficient to find optimum result in a favourable manner.

Table 9: Summary of the comparisons

L_{gs}	$R(L_{gs})$	m_{gs}	Comparison	Joined entire price for multi-item in multi-outlet distribution network system
3	53.2	3	Crisp multi-item in multi-outlet distribution network inventory system	380630
4	18.2	4		
6	1.4	5	Fuzzy multi-item in multi-outlet distribution network inventory system	376927
8	0	5		
			Savings (%)	0.97

7. Conclusion

Multi-item in multi-outlet distribution network integrated supply chain system along lead time with carbon discharge price is considered for fuzzy and crisp circumstances. In the uncertain circumstances, entirely interconnected inventory factors and decision variables are assumed to be hexagonal fuzzy quantities. Among defuzzification, the alpha cut technique is employed for the evaluation of minimum joined entire price for the multi-item in multi-outlet distribution network system. The addition of Lagrangian process is employed to achieve each item's optimum order size for each outlet. A computational algorithm is made use for the investigation of special results for fuzzy inputs on minimum joined entire price of whole system, multi-item in multi-outlet distribution network. Each item's optimum order size for each outlet is constructed on recommended inventory system. Computational efficiency is maximized by automating the calculation of total network inventory costs, which effectively bridges the gap between crisp operational data and projected fuzzy variables through iterative refinement. This structured approach ensures that complex multi-outlet variables are synchronized within the MATLAB environment for optimum accuracy. Graphical representation demonstrates that as fuzziness increases, our model maintains a lower total cost than the crisp model. An exclusive size of funds in a multi-item in multi-outlet distribution network integrated supply chain system is found. Consequently, by relating crisp and fuzzy systems, it is perceived that the multi-item in multi-outlet distribution network of the fuzzy system is enhanced than the crisp system. A comparative analysis reveals that the fuzzy logic framework demonstrates better performance and precision over the traditional crisp model. The evaluation shows 0.97% cost saving under the multi-item in multi-outlet distribution network of the fuzzy uncertainty scenarios.

Future research may extend this framework by incorporating practical limitations such as warehouse capacity, fixed setup expenditures, and procurement ceilings. Additionally, the logic can be adapted to evaluate various multi-tier supply chain configurations across crisp, fuzzy, or hybrid environments.

Authors Contribution

Authors have contributed equally to this work.

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