



Whole Domination Parameter in Zero-Divisor Graphs with Applications to Minimal Biomarker Identification

Christy P Antony* and Shiny Jose†

Abstract

Domination theory is instrumental in the structural analysis of graphs and their applications to real-world systems. In this article, we develop and investigate whole domination parameter in zero-divisor graphs derived from commutative rings. We construct characterization for whole dominating sets, and recognize algebraic criteria governing the existence or non-existence of such sets. Also whole domination number of zero-divisor graph structures assigned to rings that are product of finite fields and rings such as Z_n are computed. Furthermore, the concept is implemented to a biological framework, where vertices represent metabolic biomarkers and edges denote pathological interactions. Through this mapping, whole domination facilitates a mathematical formulation for identifying minimal sets of biomarkers that can monitor or predict metabolic dysfunction. The study bridges algebraic graph theory and biomedical modelling, offering both a new theoretical perspective and a potential diagnostic interpretation.

Keywords: Zero-divisor graph; Whole domination; Whole dominating set; Whole domination number; Biomarker identification.

AMS Subject Classification: 05C25, 05C69, 92C42.

1. Introduction

The analysis of graph structures derived from commutative rings has evolved as a significant research area through a systematic synthesis of algebraic models and graph-theoretic concepts. One of the earliest and highly impactful contributions in this direction was made by Beck[6] in 1988, who proposed the framework for associating a graph to a commutative ring.

* Department of Mathematics, Christ College, Puliyanmala, Kattappana P.O. 685515, Kerala, India; antonychristyp@gmail.com

† Department of Mathematics, St. George's College, Aruvithura P.O. 686122, Kottayam, Kerala, India; shinyjosedavis@gmail.com

Beck defined the structure using all elements of the ring as vertices and was directed towards graph coloring, thereby laying the conceptual foundation for illustrating algebraic interactions through graphs. A significant modification of this idea was subsequently provided by Anderson and Livingston[4] in 1999, who defined the vertex set using the non-zero zero-divisors of a commutative ring, and adjacency between two distinct vertices are defined if and only if their product is zero. This construction established a clear and elegant correspondence between algebraic annihilation and graph adjacency. Since then, zero-divisor graphs have been investigated extensively, with researchers [1], [3] exploring their structural properties including connectivity, diameter, girth, clique number, chromatic number, and planarity, thereby strengthening the bridge between commutative algebra and graph theory.

As the theory matured, attention shifted toward domination-related parameters in zero-divisor graphs. Domination is among the most fundamental and widely explored concept in the field of graph theory because of its strong theoretical structure and wide spectrum of applications. Several researchers [9] investigated domination numbers and related parameters in zero-divisor graphs, revealing how algebraic properties of rings influence domination structures. Building upon this foundation, the theory of whole domination proposed by A.A. Omran and Ibrahim[8] attracted interest due to its stricter domination requirements. A subset $D \subseteq V(G)$ is called a whole dominating set (WDS) if every vertex in $V - D$ is adjacent to all vertices in D . The minimum cardinality of a minimal whole dominating set (MWDS) is called the whole domination number, denoted by $\gamma_{wh}(G)$. This parameter provides a more robust and globally comprehensive notion of influence than classical domination, since each vertex outside D must be simultaneously connected to all vertices in D . Omran and Ibrahim established several foundational properties of whole domination, including bounds on $\gamma_{wh}(G)$, its behavior under graph operations such as union, join, Cartesian product, vertex deletion, and edge contraction, and conditions for its existence. Inspired by this parameter, the notion of whole domination was extended to zero-divisor graphs, adapting the classical definition to the algebraic setting.

In this research, we investigate the whole domination parameter in zero-divisor graphs and explore its algebraic and structural implications.

Bounds and characterizations of $\gamma_{wh}(\Gamma(R))$ are obtained for various classes of commutative rings. Furthermore, the framework is extended towards a biomedical interpretation[10], [11], where zero-divisor graphs are used to model biomolecular or diagnostic systems. The whole domination number corresponds to identifying a minimal biomarker set that completely influences or represents all other biomarkers in a biological network. Such

an interpretation highlights the relevance of algebraic graph parameters in the context of minimal biomarker identification, providing a theoretical foundation for feature reduction and disease classification. The proposed study thus combines the algebraic richness of zero-divisor graph structures with the depth of whole domination theory, leading to both new mathematical insights and potential applications in biomedical data analysis.

2. Preliminaries

Definition 2.1. [4]: Let R be a commutative ring and $Z^*(R)$ denote its set of non-zero zero-divisors. The zero divisor graph, $\Gamma(R)$ is a graph with vertex set $Z^*(R)$ and an edge between x and y if and only if $xy=0$ in R .

Definition 2.2. [8]: For a simple connected graph G , a proper subset $D \subset V$ is called WDS, if every vertex in set $V - D$ is adjacent to all vertices in set D . The set D is called minimal WDS (MWDS) if it has no proper WDS. The whole domination number denoted by $\gamma_{wh}(G)$ is the minimum cardinality of a MWDS. The MWDS that has γ_{wh} is called γ_{wh} - set.

Proposition 2.1. [8]:

1. $\gamma_{wh}(S_n) = 1, n \geq 2$
2. $\gamma_{wh}(C_n) = 1$ if $n = 3$ and $\gamma_{wh}(C_n) = 2$ if $n = 4$, otherwise C_n has no WDS.
3. $\gamma_{wh}(P_n) = 1$ if $n = 2, 3$, otherwise P_n has no WDS.
4. $\gamma_{wh}(K_{m,n}) = m, 2 \leq m \leq n$.

3. Whole domination in zero-divisor graph

Definition 3.1. A zero-divisor graph $\Gamma(R)$ is said to possess a whole dominating set if there exists a nonempty proper subset $D \subset Z^*(R)$ such that every vertex $v \in Z^*(R) \setminus D$ is adjacent to all vertices of D , i.e., $\forall v \in Z^*(R) \setminus D, \forall d \in D, vd = 0$. Equivalently, $\Gamma(R)$ contains a spanning complete bipartite subgraph $K_{|D|, |Z^*(R) \setminus D|}$.

Theorem 3.1: The zero-divisor graph $\Gamma(R)$ has a whole dominating set if and only if the set of non-zero zero-divisors $Z^*(R)$ can be partitioned into two non-empty disjoint subsets $Z^*(R) = X \cup Y$ such that $xy=0$ for all $x \in X$ and $y \in Y$. In this case, either X or Y serves as a whole dominating set of $\Gamma(R)$.

Proof. Consider the whole dominating set D of $\Gamma(R)$. Then every vertex in $Z^*(R) \setminus D$ is adjacent to every vertex in D , implying $xy=0$ for all $x \in D$ and $y \in Z^*(R) \setminus D$. Setting $X=D$ and $Y = Z^*(R) \setminus D$, we have $Z^*(R) = X \cup Y$ with $xy=0$ for all $x \in X, y \in Y$. Conversely, suppose such a partition $Z^*(R) = X \cup Y$ with $xy=0$ for all $x \in X, y \in Y$. Then each vertex in Y is adjacent to every vertex in X . Hence, X is a whole dominating set of $\Gamma(R)$.

Corollary 3.1. If $\Gamma(R)$ possesses a whole dominating set, then $\Gamma(R)$ contains a spanning complete bipartite subgraph and consequently has diameter two.

Corollary 3.2. Let $D \subseteq Z^*(R)$. Then D is a whole dominating set of $\Gamma(R)$ if and only if

$$Z^*(R) \setminus D \subseteq \bigcap_{d \in D} \text{Ann}(d)$$

where $\text{Ann}(d) = \{ r \in R \mid rd = 0 \}$.

Corollary 3.3. If $R \cong R_1 \times R_2$, where $|R_1|, |R_2| \geq 2$, then

$$Z^*(R) = \{(a,0) : a \in R_1 \setminus \{0\}\} \cup \{(0,b) : b \in R_2 \setminus \{0\}\}$$

and every element of one set annihilates every element of the other. Hence,

$$\Gamma(R) \cong K_{|R_1|-1, |R_2|-1} \text{ and } \gamma_{wh}(\Gamma(R)) = \min\{|R_1| - 1, |R_2| - 1\}.$$

Corollary 3.4. If there exist nonzero ideals I and J of R such that $IJ=0$ and $Z^*(R) = (I \setminus \{0\}) \cup (J \setminus \{0\})$, then $\Gamma(R)$ admits a whole dominating set.

Theorem 3.2. The graph $\Gamma(R)$ does not admit a whole dominating set if and only if for every nonempty proper subset $X \subset Z^*(R)$, there exist elements $x \in X$ and $y \in Z^*(R) \setminus X$ such that $xy \neq 0$.

Proof. This is the logical negation of the partition characterization of WDS. By the characterization of whole dominating sets, $\Gamma(R)$ admits a WDS precisely when there exists a partition $Z^*(R) = X \cup Y$ with $xy=0$ for all $x \in X, y \in Y$. Hence $\Gamma(R)$ has no WDS exactly when no such nonempty proper part A exists. Equivalently, for every nonempty proper $X \subset Z^*(R)$ there must be some cross pair $(x,y) \in X \times (Z^*(R) \setminus X)$ with $xy \neq 0$.

Corollary 3.5. Each of the following conditions implies $\Gamma(R)$ has no whole dominating set:

Large diameter: If $\text{diam}(\Gamma(R)) > 2$, then $\Gamma(R)$ cannot have a WDS (since any WDS produces a spanning complete bipartite subgraph and forces $\text{diam} \leq 2$).

Non-existence of annihilating ideal cover: If there do not exist non-zero ideals $I, J \subseteq R$ with $IJ = 0$ and $Z^*(R) = (I \setminus \{0\}) \cup (J \setminus \{0\})$, then $\Gamma(R)$ has no WDS.

Theorem 3.3. For the zero-divisor graph $\Gamma(R)$, the listed statements are equivalent:

1. $\Gamma(R)$ does not possess any whole dominating set.
2. $\bigcap_{x \in Z^*(R)} \text{Ann}(x) = \{0\}$ (i.e. there is no non-zero element that annihilates all non-zero zero-divisors).
3. $\Gamma(R)$ contains no spanning complete bipartite subgraph $K_{m,n-m}$.

Proof. (1) \Leftrightarrow (2). By definition (and as used in the characterization provided earlier), D is a WDS of $\Gamma(R)$ iff every $d \in D$ satisfies $vd = 0$ for all

$v \in Z^*(R) \setminus D$. Hence the existence of a nonempty WDS is equivalent to the existence of a non-zero $a \in R$ with $ab=0$ for every $b \in Z^*(R)$, i.e. $\bigcap_{b \in Z^*(R)} \text{Ann}(b) \neq \{0\}$. The negation of that statement is (2).

(1) \Leftrightarrow (3). According to Remark 2.1(ii) of [5], a graph G admits a WDS iff G has a spanning complete bipartite subgraph. Applying that to $\Gamma(R)$ gives the equivalence between (1) and (3).

Corollary 3.6. If any of the following holds, then $\Gamma(R)$ does not have a WDS:

- (1) $\Gamma(R)$ is disconnected.
- (2) $\text{diam}(\Gamma(R)) \geq 3$.
- (3) There is no non-zero $a \in R$ with $ab = 0$ for all $b \in Z^*(R)$.

4. Whole domination in some zero-divisor graphs

In this section, we obtain the values of whole domination parameter for zero-divisor graph of rings of integer modulo n and rings that are product of finite fields.

Theorem 4.1. For any ring $R = F_m \times F_n$, where F_m and F_n are finite fields of order m and n respectively, $\gamma_{wh}(\Gamma(R)) = \min\{m-1, n-1\}$

Proof. The vertex set of the graph $\Gamma(R)$ is defined as $V = \{(v_1, 0), (v_2, 0), \dots, (v_{m-1}, 0), (0, u_1), (0, u_2), \dots, (0, u_{n-1})\}$. The vertex set can be partitioned into two disjoint subsets $V_1 = \{(v_1, 0), (v_2, 0), \dots, (v_{m-1}, 0)\}$ and $V_2 = \{(0, u_1), (0, u_2), \dots, (0, u_{n-1})\}$ such that every vertex of V_1 is connected to every vertex of V_2 and no edge has endpoints in the same set. Hence $\Gamma(R)$ is isomorphic to $K_{m-1, n-1}$. Therefore $\gamma_{wh}(\Gamma(R)) = \gamma_{wh}(K_{m-1, n-1}) = \min\{m-1, n-1\}$.

Corollary 4.1. For any graph $\Gamma(Z_{pq})$, where p and q are prime numbers with $p < q$, $\gamma_{wh}(\Gamma(Z_{pq})) = p-1$

Theorem 4.2. For an integral domain R , $\gamma_{wh}(\Gamma(Z_2 \times R)) = 1$.

Proof. Clearly $\Gamma(Z_2 \times R)$ is a star graph if R is an integral domain. Hence $\gamma_{wh}(\Gamma(Z_2 \times R)) = 1$.

Theorem 4.3. Let $p \geq 3$ be a prime. Then $\gamma_{wh}(\Gamma(Z_2 \times Z_p)) = 1$.

Proof. Since Z_p is a field, it has no non-zero zero-divisors. Thus the non-zero zero-divisors of $Z_2 \times Z_p$ are exactly the elements of the form $(0, b)$ with $b \in Z_p^*$ and the elements of the form $(1, 0)$. Hence $V(\Gamma(Z_2 \times Z_p)) = \{(0, 1), \dots, (0, p-1), (1, 0)\}$. For any $(0, b)$, we have $(1, 0) \cdot (0, b) = (0, 0)$, so $(1, 0)$ is adjacent to every $(0, b)$. Therefore $\{(1, 0)\}$ is a whole dominating set which makes $\gamma_{wh} = 1$.

Theorem 4.4. For any graph $\Gamma(Z_{p^2})$, where p is any prime number, $\gamma_{wh}(\Gamma(Z_{p^2})) = 1$.

Proof. The vertex set of the graph $\Gamma(Z_{p^2})$ is defined as $V = \{p, 2p, \dots, p(p-1)\}$. Since any two vertices in V are adjacent, $\Gamma(Z_{p^2})$ is a complete graph and is isomorphic to K_{p-1} . Hence $\gamma_{wh}(\Gamma(Z_{p^2})) = 1$.

Theorem 4.5. For any graph $\Gamma(Z_{p^n})$, where $n > 1$ and p is any odd prime number, $\gamma_{wh}(\Gamma(Z_{p^n})) = 1$.

Proof. For the vertex set $V = \{p, 2p, 3p, \dots, p^{n-1}, p(p^{n-1}-1)\}$ of $\Gamma(Z_{p^n})$, the vertex p^{n-1} is adjacent to all other vertices in V and $\{p^{n-1}\}$ forms γ_{wh} set. Hence $\gamma_{wh}(\Gamma(Z_{p^n})) = 1$.

Corollary 4.2. For any graph $\Gamma(Z_{2^n})$, where $n \geq 3$, $\gamma_{wh}(\Gamma(Z_{2^n})) = 1$.

Proof. Consider the vertex set $V = \{2, 4, 6, \dots, 2^{n-1}, 2(2^{n-1}-1)\}$. Since the vertex 2^{n-1} is adjacent to all other vertices in V , $\{2^{n-1}\}$ forms γ_{wh} set. Hence $\gamma_{wh}(\Gamma(Z_{2^n})) = 1$.

Theorem 4.6. If $\Gamma(R)$ is a tree, then it possesses a whole dominating set if and only if it is a star graph. Consequently, $\gamma_{wh}(\Gamma(R)) = 1$.

Proof. Since $\Gamma(R)$ is a tree, the result follows directly from Proposition 2.6 of [8].

5. Application of Whole Domination in Zero-Divisor Graphs for Minimal Biomarker Identification.

In clinical diagnosis, complex biological or clinical systems can be represented as networks of interacting parameters. Each vertex of the graph may represent a physiological parameter, biochemical marker, or clinical symptom, while an edge depicts an interaction or dependency between two such parameters. When two parameters together lead to a failure or neutralization of normal function, this can be viewed as a zero-divisor interaction. Hence, the zero-divisor graph provides a concise representation of the interplay among clinical factors. A whole dominating set in this context aligns with a minimal subset of critical biomarkers or symptoms that influence or reflect all other parameters in the diagnostic network. Thus, whole domination number represents the smallest number of parameters that need to be monitored to ensure complete diagnostic coverage.

5.1. Illustration: Z_{12} Model for Metabolic Diagnosis

Consider the ring Z_{12} , whose vertex set is $Z^*(Z_{12}) = \{2, 3, 4, 6, 8, 9, 10\}$. The zero-divisor graph $\Gamma(Z_{12})$ is generated by joining two vertices a and b whenever $ab \equiv 0 \pmod{12}$. For instance, in a metabolic disorder model, the mapping can be conceptualized as:

2: Glucose metabolism

3: Insulin sensitivity

- 4 : Liver enzyme function
 6 : Cholesterol synthesis
 8 : Oxidative stress marker
 9 : Inflammatory response
 10 : Kidney filtration

The following table gives the reasoning behind the edges in the metabolic disorder model based on $\Gamma(Z_{12})$.

| Edge (a, b) | Biological reasoning |
|-------------|--|
| (2,6) | Glucose metabolism and Cholesterol synthesis are strongly coupled in hepatic regulation. Excess glucose enhances lipid and cholesterol synthesis via insulin-dependent pathways, leading to dyslipidaemia. |
| (3, 6) | Reduced Insulin sensitivity increases hepatic lipid output, resulting in elevated cholesterol and promoting metabolic syndrome |
| (4, 6) | Liver enzyme function directly influences cholesterol synthesis; hepatic dysfunction disrupts enzymatic lipid control, causing abnormal cholesterol accumulation. |
| (8, 6) | Oxidative stress oxidises LDL cholesterol, disturbing lipid balance and accelerating atherosclerotic changes. |
| (9, 2) | The Inflammatory response interferes with Glucose metabolism by inducing insulin resistance, resulting in hyperglycaemia |
| (9, 3) | Chronic inflammation reduces insulin sensitivity, forming a key mechanism in Type 2 diabetes. |
| (9, 4) | Inflammation elevates hepatic enzymes, causing hepatic stress and disrupting detoxification. |
| (9, 10) | Inflammatory mediators impair kidney filtration, leading to renal tissue damage |
| (6, 9) | Cholesterol synthesis and inflammation exhibit bidirectional influence: high cholesterol induces vascular inflammation, while inflammation stimulates cholesterol accumulation in macrophages. |

Table 1: Interpretation of edges in $\Gamma(Z_{12})$

An edge such as (2,6), where $2 \cdot 6 \equiv 0 \pmod{12}$, can thus be interpreted as representing the interaction between glucose metabolism and cholesterol synthesis, whose joint dysfunction indicates a metabolic shutdown stage or zero functional efficiency. Therefore, the algebraic zero corresponds to a clinically critical state or a medical unstable condition.

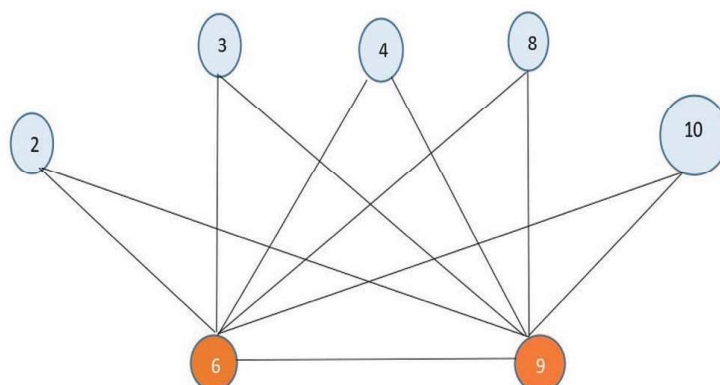


Figure 1: Zero-divisor graph $\Gamma(Z_{12})$ showing a whole dominating set $\{6,9\}$

A possible whole dominating set is $D = \{6,9\}$, implying that every other vertex in the graph is adjacent to both 6 and 9. The vertices 6 and 9 can be perceived as key physiological parameters whose monitoring suffices to infer the comprehensive system status. Since every other parameter is connected to both cholesterol synthesis and inflammatory response, these two biomarkers together provide an extensive overview of the disease state. Therefore, $\gamma_{wh}(\Gamma(Z_{12})) = 2$ can be interpreted as the minimum diagnostic coverage number.

5.2. Applications in Medical Diagnosis

The concept of whole domination in zero-divisor graphs can be used to:

- (1) Identify minimal sets of biomarkers sufficient for diagnosis.
- (2) Reduce duplication in large-scale clinical testing.
- (3) Model symptom interdependencies and identify the most influential physiological parameters.
- (4) Facilitate drug target discovery by locating minimal sets of biological factors influencing the entire metabolic network.

Hence, the mathematical notion of whole domination in $\Gamma(R)$ provides a theoretical framework for selecting efficient, comprehensive, and cost-effective diagnostic indicators.

6. Conclusion

In this study, an exhaustive analysis of the whole domination number of zero-divisor graphs linked to diverse classes of commutative rings were provided. The results established a clear correlation between the diameter of the zero-divisor graph and its whole dominating capacity. By examining the parameter across different algebraic contexts, we have demonstrated how the internal multiplicative structure of a ring dictates its domination capacity.

Furthermore, the practical utility of this parameter was showcased through its application in minimal biomarker identification in clinical diagnosis.

References

- [1]. M. Afkhami and K. Khashyarmansh, On the diameter of zero-divisor graphs, *Acta Mathematica Hungarica*, 122 (2009), 1–8.
- [2]. Alsinai, A. A. Omran and H. A. Oathman, Some properties of whole edge domination in graphs, *Appl. Math. Inf. Sci.*, **16**(3) (2022), 435–440. <https://doi.org/10.18576/amis/160303>.
- [3]. S. Akbari and A. Mohammadian, On the zero-divisor graph of a commutative ring, *Journal of Algebra*, 296 (2006), 462–479.
- [4]. D. F. Anderson and P. S. Livingston, The zero-divisor graph of a commutative ring, *Journal of Algebra*, 217 (1999), 434–447. <https://doi.org/10.1006/jabr.1998.7840>.
- [5]. M. F. Atiyah and I. G. Macdonald, *Introduction to Commutative Algebra* Addison-Wesley, Reading, MA, (1969).
- [6]. I. Beck, Coloring of commutative rings, *Journal of Algebra*, **116** (1988), 208–226. [https://doi.org/10.1016/0021-8693\(88\)90202-5](https://doi.org/10.1016/0021-8693(88)90202-5).
- [7]. T. W. Haynes, S. T. Hedetniemi and P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, USA (1998).
- [8]. A. A. Omran and T. A. Ibrahim, Whole domination in graph, *TWMS J. Appl. Eng. Math.*, 12 (2022), 1506–1511.
- [9]. N. J. Rad, S. H. Jafari and D. A. Mojdeh, On Domination in Zero-Divisor Graphs, *Canadian Mathematical Bulletin*, 56(2) (2013), 407–411.
- [10]. H. Zheng, C. Gu and H. Yang, Identification of disease-specific bio-markers through network-based analysis of gene co-expression: A case study on Alzheimer’s disease, *Heliyon* 10(5) (2024). <https://doi.org/10.1016/j.heliyon.2024.e27070>
- [11]. C. Zoccali *et.al.*, Biomarkers in clinical epidemiology studies, *Clinical Kidney Journal*, 17(6) (2024). <https://doi.org/10.1093/ckj/sfae130>