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$Gg\beta^*$ - Compactness in Grill Topological Spaces

P. Gomathi Rajakumari*, Jessie Theodore†

Abstract

In this paper, we have introduced and investigated the characterizations of $Gg\beta^*$ -compact space in Grill Topological Spaces. Also compact modulo- $Gg\beta^*$ -space and countably - $Gg\beta^*$ -compact space are introduced. Some of the equivalent conditions and properties of these spaces are discussed, specially pointing on their characterization under $Gg\beta^*$ -continuous function and $Gg\beta^*$ -open covers. The relationship between the new spaces are discovered.

Keywords: $Gg\beta^*$ -closed set, $Gg\beta^*$ -open cover, $Gg\beta^*$ -compact, $Gg\beta^*$ -continuous function, Grill Topological Spaces

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1. Introduction

In [2] Choquet introduced the concept of Grills on a Topological Space. The concept of grills has shown to be a powerful supporting and useful tool like nets and filters, for getting a deeper insight into further studying some topological notions such as proximity spaces, closure spaces and the theory of compactifications and extension problems of different kinds. Karthika A and Prathiba S R [7] discovered α -compactness and examined the properties of α -compactness in Grill Topological Spaces. In this paper, we have introduced new type of compact spaces such as $Gg\beta^*$ - compactness, compact modulo-$Gg\beta^*$-space and countably-$Gg\beta^*$ -compact space. Some of the equivalent conditions and properties of these spaces are discussed, specially pointing on their characterization under $Gg\beta^*$-continuous function and $Gg\beta^*$ -open covers. The relationship between the new spaces are discovered.

* Department of Mathematics and Research Centre, Sarah Tucker College, Tirunelveli - 7. (Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli -12, Tamil Nadu, India); grkumari9495@gmail.com; jessielawrence2703@gmail.com

2. Preliminaries

2.1 Definition: [2] A non-empty collection G is called Grill on $\mathbb{X}$ if

(i) $\phi \notin G$

(ii) $E \in G\ and\ E \subseteq H \Longrightarrow H \in G,$

(iii) $E, H \subseteq \mathbb{X}\ and\ E \cup H \in G \Longrightarrow E \in G\ or\ H \in G.$

2.2 Definition: [4] Let a Grill Topological Space be $(\mathbb{X}, \tau, G)$. A $\Phi_{G\beta}$: P($\mathbb{X}$) → P($\mathbb{X}$) is a mapping defined by $\Phi_{G\beta}(\mathcal{L}) = \{x \in \mathbb{X} : U \cap \mathcal{L} \in G, \forall\ U \in G\beta 0(x)\}$ for each $\mathcal{L} \in P(\mathbb{X})$. Then $\Phi_{G\beta}$ is called the Gβ-operator and $G\beta 0(x) = \{U \in G\beta 0(\mathbb{X}) : x \in U\}$. The Gβ-closure operator $\Psi_{G\beta}$: P($\mathbb{X}$) → P($\mathbb{X}$) is defined by $\Psi_{G\beta}(\mathcal{L}) = \mathcal{L} \cup \Phi_{G\beta}(\mathcal{L})$ for all $\mathcal{L} \in P(\mathbb{X})$.

2.3 Definition: [4] If $\Phi_{G\beta}(\mathcal{L}) \subseteq$ int U whenever $\mathcal{L} \subseteq$ U and U is ω open in $(\mathbb{X}, \tau)$, then the set $\mathcal{L}$ of $\mathbb{X}$ is Ggβ* closed in $(\mathbb{X}, \tau, G)$. The family of all Ggβ* closed sets are denoted by Ggβ* $C(\mathbb{X}, \tau)$.

2.4 Proposition: [4] Every G- semi closed (respectively. Gα –closed, G-closed, β^*-closed, *g- closed, g*- closed, $g^{\#}s$- closed, $g^{\#}$- closed, Ggb closed, Gg* closed, b^*g closed) sets are Ggβ*-closed set but not conversely.

3. $\boldsymbol{Gg\beta^*}$ - Compactness

3.1 Definition: In Grill Topological Space $(\mathbb{X}, \tau, G)$, a collection $\{\mathcal{H}_i : i \in \Lambda\}$ made of $Gg\beta^*$ -open sets and $\mathcal{H} \subseteq \bigcup_{i\in\Lambda}\mathcal{H}_i$, then the collection is said to be $Gg\beta^*$-open cover for a subset $\mathcal{H}$ in $(\mathbb{X}, \tau, G)$.

3.2 Definition: If every $Gg\beta^*$-open cover of $(\mathbb{X}, \tau, G)$ has a finite subcover, then the Grill Topological Space $(\mathbb{X}, \tau, G)$ is said to be $Gg\beta^*$- compact.

3.3 Theorem: In a $Gg\beta^*$- compact space, a $Gg\beta^*$-closed subset is $Gg\beta^*$-compact.

Proof: Let $\mathcal{H}$ be a $Gg\beta*$-closed subset of a $Gg\beta*$-compact space $(\mathbb{X}, \tau, G)$. Then $\mathcal{H}^c$ is a $Gg\beta^*$-open. Assume that a $Gg\beta^*$-open cover which covers $\mathcal{H}$ in $\mathbb{X}$. Suppose this $Gg\beta^*$-open cover along with $\mathcal{H}^c$ form a $Gg\beta^*$ -open cover of $\mathbb{X}$. Assume that $\{w_1, w_2, \ldots, w_n\}$is a finite subcover of $\mathbb{X}$, because $(\mathbb{X}, \tau, G)$ is $Gg\beta^*$-compact. Suppose $\mathcal{H}^c$ in this subcover, we discard it. Otherwise depart from the subcover as it is. Finally, we have obtained a finite subcover of $\mathcal{H}$. This implies that $\mathcal{H}$ is $Gg\beta^*$-compact.

3.4 Corollary: Every closed (G- closed, Gα - closed and hence G semi closed) subset of a $Gg\beta^*$-compact space is $Gg\beta^*$- compact.

Proof: by proposition 2.4 and theorem 3.3

3.5 Remark: In a Grill Topological Space $(\mathbb{X}, \tau, G)$,

(i) Every Compact space $(\mathbb{X}, \tau)$ is $Gg\beta^*$-compact space for any Grill G on $\mathbb{X}$.

(ii) If the space $(\mathbb{X}, \tau_{G\beta})$ is compact space, then $(\mathbb{X}, \tau)$ is compact (as $\tau \subseteq \tau_{G\beta}$) and hence $(\mathbb{X}, \tau)$ is $Gg\beta^*$ - compact.

(iii) If $(\mathbb{X}, \tau_{G\beta})$ is $Gg\beta^*$ - compact, then $(\mathbb{X}, \tau)$ is $Gg\beta^*$ - compact (as $\tau \subseteq \tau_{G\beta}$).

$$\begin{array}{ccc} (\mathbb{X}, \tau) \text{ is compact} & \Longleftarrow & (\mathbb{X}, \tau_{G\beta}) \text{ is compact} \\ \Downarrow & & \Downarrow \\ (\mathbb{X}, \tau) \text{ is } Gg\beta^*\text{-compact} & \Longleftrightarrow & (\mathbb{X}, \tau_{G\beta}) \text{ is } Gg\beta^*\text{-compact} \end{array}$$

Figure 3.1: Comparison between compact and $Gg\beta^*$-compact

3.6 Theorem: Let a Grill Topological Space be $(\mathbb{X}, \tau, G)$. $(\mathbb{X}, \tau, G)$ is a $Gg\beta^*$-compact space if and only if for every family $\{U_\alpha: \alpha \in \Lambda\}$ which is made by $Gg\beta^*$- closed sets such that $\bigcap\{U_\alpha: \alpha \in \Lambda\} = \phi$, there exists a finite subset Λ_0 of Λ such that $\bigcap\{U_\alpha: \alpha \in \Lambda_0\} = \phi$.

Proof:

Assume that $\{U_\alpha: \alpha \in \Lambda\}$ is a family of $Gg\beta^*$-closed sets such that $\bigcap\{U_\alpha: \alpha \in \Lambda\} = \phi$. If $\bigcap\{U_\alpha: \alpha \in \Lambda\} = \phi$, then $\bigcup\{U_\alpha^c: \alpha \in \Lambda\} = \mathbb{X}$. Hence, $\bigcup\{U_\alpha^c: \alpha \in \Lambda\}$ is a covering of $\mathbb{X}$ by $Gg\beta^*$-open sets. By hypothesis, there exists a finite subset Λ_0 of Λ such that $\bigcup\{U_\alpha^c: \alpha \in \Lambda_0\} = X$, that is $\bigcap\{U_\alpha: \alpha \in \Lambda_0\} = \phi$.

Conversely, assume that $\{U_\alpha: \alpha \in \Lambda\}$ be a cover of $\mathbb{X}$ by $Gg\beta^*$-open sets. Then $\mathbb{X} = \bigcup\{U_\alpha: \alpha \in \Lambda\}$, implies that $\bigcap\{U_\alpha^c: \alpha \in \Lambda\} = \phi$. By hypothesis, there exists a Λ_0 which is finite subset of Λ such that $\bigcap\{U_\alpha^c: \alpha \in \Lambda_0\} = \phi$, that is $\bigcup\{U_\alpha: \alpha \in \Lambda_0\} = \mathbb{X}$. Hence $(\mathbb{X}, \tau, G)$ is a $Gg\beta^*$-compact space.

3.7 Definition: Let $(\mathbb{X}, \tau, G_1)$ and $(\mathbb{Y}, \sigma, G_2)$ be two Grill Topological Space. A function $f: (\mathbb{X}, \tau, G_1) \to (\mathbb{Y}, \sigma, G_2)$ is called

(i) $Gg\beta^*$ - open function if $f(A)$ is $Gg\beta^*$ - open set in $\mathbb{Y}$ for each $Gg\beta^*$ - open set in $\mathbb{X}$.

(ii) $Gg\beta^*$ - continuous function if $f^{-1}(A)$ is $Gg\beta^*$ - open set in $\mathbb{X}$ for each open set A in $\mathbb{Y}$.

(iii) $Gg\beta^*$ - irresolute function if $f^{-1}(A)$ is $Gg\beta^*$ - open set in $\mathbb{X}$ for each $Gg\beta^*$ - open set A in $\mathbb{Y}$.

(iv) strongly-$Gg\beta^*$ -continuous if the inverse image of every $Gg\beta^*$ -open set of $\mathbb{Y}$ is open in $\mathbb{X}$.

3.8 Theorem: In surjective $Gg\beta^*$ -continuous map, an image of a $Gg\beta^*$ -compact space is compact.

Proof: Let $f: (\mathbb{X}, \tau, G_1) \to (\mathbb{Y}, \sigma, G_2)$ be a $Gg\beta^*$-continuous onto map and $(\mathbb{X}, \tau, G_1)$ be $Gg\beta^*$-compact. Assume that an open cover of $\mathbb{Y}$ is $\{U_\alpha: \alpha \in \Lambda\}$. Then $\{f^{-1}(U_\alpha): \alpha \in \Lambda\}$ is a $Gg\beta^*$-open cover of $\mathbb{X}$. Since $(\mathbb{X}, \tau, G_1)$ is $Gg\beta^*$-compact, then the finite subcover is $\{f^{-1}(U_1), f^{-1}(U_2), \dots, f^{-1}(U_n)\}$. Hence $\{U_1, U_2, \dots, U_n\}$ is an open cover of $\mathbb{Y}$ because f is onto. This implies that $(\mathbb{Y}, \sigma, G_2)$ is compact.

3.9 Theorem: In a strongly-$Gg\beta^*$-continuous onto map $f: (\mathbb{X}, \tau, G_1) \to (\mathbb{Y}, \sigma, G_2)$, $(\mathbb{Y}, \sigma, G_2)$ is $Gg\beta^*$-compact if $(\mathbb{X}, \tau, G_1)$ is a compact space.

Proof: Let $\{U_\alpha: \alpha \in \Lambda\}$ be a $Gg\beta^*$-open cover of $(\mathbb{Y}, \sigma, G_2)$. Since f is strongly $Gg\beta^*$-continuous, $\{f^{-1}(U_\alpha): \alpha \in \Lambda\}$ is an open cover of $(\mathbb{X}, \tau, G_1)$. Here $(\mathbb{X}, \tau, G_1)$ is a compact, it has a finite subcover say, $\{f^{-1}(U_1), f^{-1}(U_2), \dots, f^{-1}(U_n)\}$. Since f is onto, $\{U_1, U_2, \dots, U_n\}$ is a finite subcover of $(\mathbb{Y}, \sigma, G_2)$ and therefore $(\mathbb{Y}, \sigma, G_2)$ is $Gg\beta^*$-compact.

3.10 Corollary: In a perfectly-$Gg\beta^*$-continuous onto map $f: (\mathbb{X}, \tau, G_1) \to (\mathbb{Y}, \sigma, G_2)$, $(\mathbb{Y}, \sigma, G_2)$ $Gg\beta^*$ -compact if $(\mathbb{X}, \tau, G_1)$ is a compact space.

Proof: Since every perfectly-$Gg\beta^*$-continuous function is strongly $Gg\beta^*$-continuous and therefore the result follows from above theorem.

3.11 Corollary: If a map $f: (\mathbb{X}, \tau, G_1) \to (\mathbb{Y}, \sigma, G_2)$ is a strongly-Gg^*-continuous onto map where $(\mathbb{X}, \tau, G_1)$ is a compact space, then $(\mathbb{Y}, \sigma, G_2)$ is $Gg\beta^*$-compact.

Proof: Since every strongly-Gg^*-continuous map is strongly-$Gg\beta^*$- continuous the proof follows from above theorem.

3.12 Theorem: If a subset W of $\mathbb{X}$ is $Gg\beta^*$-compact relative to $\mathbb{X}$ in a $Gg\beta^*$-irresolute map $f: (\mathbb{X}, \tau, G_1) \to (\mathbb{Y}, \sigma, G_2)$, then the image $f(W)$ is $Gg\beta^*$-compact relative to $\mathbb{Y}$.

Proof: Let $Gg\beta^*$-open cover of $\mathbb{Y}$ be $\{V_\alpha: \alpha \in \Lambda\}$ such that $f(W) \subset \bigcup_{\alpha\in\Lambda} V_\alpha$. Then $W \subset \bigcup_{\alpha\in\Lambda} f^{-1}(V_\alpha)$ for each $f^{-1}(V_\alpha)$ is $Gg\beta^*$- open in $\mathbb{X}$. There exists a finite subset Λ_n of Λ such that $W \subset \bigcup_{\alpha\in\Lambda_n} f^{-1}(V\alpha)$ by hypothesis. Hence $f(W) \subset \bigcup_{\alpha\in\Lambda_n} V_\alpha$ and so $f(W)$ is $Gg\beta^*$- compact relative to $\mathbb{Y}$.

4. Compact Modulo- $\boldsymbol{Gg\beta^*}$-Space

4.1 Definition: In a Grill Topological Space $(\mathbb{X}, \tau, \mathrm{G})$, a subset $\mathcal{L}$ is called compact modulo- $Gg\beta^*$-set (or compatible-$Gg\beta^*$-compact set) if every $Gg\beta^*$-open cover $\{\mathcal{H}_\alpha: \alpha \in \Lambda\}$ of $\mathcal{L}$, there exists a finite subset Λ_0 of Λ such that $\mathcal{L} - \bigcup\{\mathcal{H}_\alpha: \alpha \in \Lambda_0\} \notin G$.

4.2 Definition: A Grill Topological Space $(\mathbb{X}, \tau, \mathrm{G})$ is said to be compact modulo-$Gg\beta^*$- space (or compatible- $Gg\beta^*$-compact space) if it is compatible-$Gg\beta^*$- compact space as a subset.

4.3 Theorem: Assume that a Grill Topological Space be $(\mathbb{X}, \tau, \mathrm{G})$. Then the following are equivalent.

(i) $(\mathbb{X}, \tau, \mathrm{G})$ is a compatible $Gg\beta^*$ - compact space.

(ii) For every family $\{\mathcal{H}_\alpha: \alpha \in \Lambda\}$ of $Gg\beta^*$ - closed sets such that $\cap\{\mathcal{H}_\alpha: \alpha \in \Lambda\} = \phi$, there exists a finite subset Λ_0 of Λ such that $\cap\{\mathcal{H}_\alpha: \alpha \in \Lambda_0\} \notin G$.

Proof:

(i) Assume that $\{\mathcal{H}_\alpha: \alpha \in \Lambda\}$ is a family of $Gg\beta^*$ - closed sets such that $\cap\{\mathcal{H}_\alpha: \alpha \in \Lambda\} = \phi$. If $\cap\{\mathcal{H}_\alpha: \alpha \in \Lambda\} = \phi$, then $\cup\{\mathcal{H}_\alpha{}^c: \alpha \in \Lambda\} = \mathbb{X}$. Hence, $\cup\{\mathcal{H}_\alpha{}^c: \alpha \in \Lambda\}$ is a covering of $\mathbb{X}$ by $Gg\beta^*$ -open sets. By hypothesis, there exists a finite subset Λ_0 of Λ such that $\mathbb{X} - \cup\{\mathcal{H}_\alpha{}^c: \alpha \in \Lambda_0\} \notin G$, that is $\cap\{\mathcal{H}_\alpha: \alpha \in \Lambda_0\} \notin G$.

(ii) Let $\{\mathcal{H}_\alpha: \alpha \in \Lambda\}$ be a $Gg\beta^*$-open cover of $\mathbb{X}$. Then $\cup\{\mathcal{H}_\alpha: \alpha \in \Lambda\} = \mathbb{X}$, implies that $\cap\{\mathcal{H}_\alpha{}^c: \alpha \in \Lambda\} = \phi$. There exists a finite subset Λ_0 of Λ such that $\cap\{\mathcal{H}_\alpha{}^c : \alpha \in \Lambda_0\} \notin G$ by hypothesis, that is $\mathbb{X} - \cup\{\mathcal{H}_\alpha: \alpha \in \Lambda_0\} \notin G$. Hence $(\mathbb{X}, \tau, \mathrm{G})$ is a compatible $Gg\beta^*$ -compact space.

4.4 Theorem: If $(\mathbb{X}, \tau, G_1)$ is compatible $Gg\beta^*$- compact space in the $Gg\beta^*$-irresolute surjective function $f: (\mathbb{X}, \tau, G_1) \to (\mathbb{Y}, \sigma, G_2)$, then $(\mathbb{Y}, \sigma, G_2)$ is so.

Proof: Let $\{\mathcal{H}_\alpha: \alpha \in \Lambda\}$ be a $Gg\beta^*$-open cover of $\mathbb{Y}$. Since f is a $Gg\beta^*$- irresolute surjective function, $\{f^{-1}(\mathcal{H}_\alpha): \alpha \in \Lambda\}$ is a $Gg\beta^*$-open cover of $\mathbb{X}$. Since $\mathbb{X}$ is compatible $Gg\beta^*$-compact, there exists a finite subset Λ_0 of Λ such that $\mathbb{X} - \cup\{f^{-1}(\mathcal{H}_\alpha): \alpha \in \Lambda_0\} \notin G_1$. This implies $\mathbb{Y} - \cup\{\mathcal{H}_\alpha: \alpha \in \Lambda_0\} = f(X - \{f^{-1}(\mathcal{H}_\alpha): \alpha \in \Lambda_0\}) \notin G_2$. Hence $\mathbb{Y}$ is compatible $Gg\beta^*$- compact space.

4.5 Theorem: If $(\mathbb{Y}, \sigma, G_2)$ is compatible $Gg\beta^*$-compact space in $Gg\beta^*$- open bijective function $f: (\mathbb{X}, \tau, G_1) \to (\mathbb{Y}, \sigma, G_2)$, then $(\mathbb{X}, \tau, G_1)$ is so.

Proof: Assume that $\{\mathcal{H}_\alpha: \alpha \in \Lambda\}$ is a $Gg\beta^*$-open cover of $\mathbb{X}$. Since f is a $Gg\beta^*$-open bijective function, $\{f(\mathcal{H}_\alpha): \alpha \in \Lambda\}$ is a $Gg\beta^*$-open cover of $\mathbb{Y}$. Since $\mathbb{Y}$ is compatible $Gg\beta^*$-compact, there exists a finite subset Λ_0 of Λ such that $\mathbb{Y} - \cup\{f(\mathcal{H}_\alpha): \alpha \in \Lambda_0\} \notin G_2$. This implies $\mathbb{X} - \cup\{\mathcal{H}_\alpha: \alpha \in \Lambda_0\} = f^{-1}(Y - \cup\{f(\mathcal{H}_\alpha): \alpha \in \Lambda_0\}) \notin G_1$. Hence $\mathbb{X}$ is compatible $Gg\beta^*$-compact space.

4.6 Remark: Every compatible $Gg\beta^*$-compact space is compatible Grill compact space but not conversely.

Proof: Obvious from the definition and $\tau \subseteq Gg\beta^*O(\mathbb{X})$.

4.7 Example: Let $\mathbb{X} = \{m, n, o, p\}$, $\tau = \{\phi, \{n, o\}, \mathbb{X}\}$ and $G = \{\{\mathfrak{m}, \mathfrak{o}\}, \{\mathfrak{n}, \mathfrak{o}\}, \{\mathfrak{m}, \mathfrak{n}, \mathfrak{o}\}, \{\mathfrak{m}, \mathfrak{o}, \mathfrak{p}\}, \{\mathfrak{n}, \mathfrak{o}, \mathfrak{p}\}, \mathbb{X}\}$. Then $Gg\beta^*O(\mathbb{X}) = P(\mathbb{X})$. Then the set $\{m, o\}$ is $Gg\beta^*$-open but not open in $(\mathbb{X}, \tau)$

4.8 Theorem: Let Grill Topological Space be $(\mathbb{X}, \tau, G)$, if $\mathcal{L}_1$, $\mathcal{L}_2$ are compact modulo $Gg\beta^*$- compact subsets relative to a space $\mathbb{X}$, then $\mathcal{L}_1 \cup \mathcal{L}_2$ is compact modulo $Gg\beta^*$- compact subset relative to a space $\mathbb{X}$.

Proof: Let $\{\mathcal{H}_\alpha: \alpha \in \Lambda\}$ be a $Gg\beta^*$- open cover of $\mathcal{L}_1 \cup \mathcal{L}_2$ of $\mathbb{X}$. Then it is a $Gg\beta^*$-open cover of both $\mathcal{L}_1$ and $\mathcal{L}_2$. Since $\mathcal{L}_1$, $\mathcal{L}_2$ are compact modulo $Gg\beta^*$- compact relative to $\mathbb{X}$, then there exists a finite subset Λ_1 and Λ_2 of Λ such that $\mathcal{L}_1 - \cup\{\mathcal{H}_\alpha: \alpha \in \Lambda_1\} \notin G$ and $\mathcal{L}_2 - \cup\{\mathcal{H}_\alpha: \alpha \in \Lambda_2\} \notin G$ respectively. Since $(\mathcal{L}_1 \cup \mathcal{L}_2) - \cup\{\mathcal{H}_\alpha: \alpha \in \Lambda_1 \cup \Lambda_2\} \subseteq (\mathcal{L}_1 - \cup\{\mathcal{H}_\alpha: \alpha \in \Lambda_1\}) \cup (\mathcal{L}_2 - \cup\{\mathcal{H}_\alpha: \alpha \in \Lambda_2\}) \notin G$. Then there exists a finite subset $\Lambda_1 \cup \Lambda_2$of Λ such that $(\mathcal{L}_1 \cup \mathcal{L}_2) - \cup\{\mathcal{H}_\alpha: \alpha \in \Lambda_1 \cup \Lambda_2\} \notin G$, thus $\mathcal{L}_1 \cup \mathcal{L}_2$ is compact modulo $Gg\beta^*$-compact subset relative to a space $\mathbb{X}$.

4.9 Theorem: In the Grill Topological Space ($\mathbb{X}$, τ, G), $\mathcal{L}$ is compact modulo $Gg\beta^*$-compact relative to $\mathbb{X}$, then $(\mathcal{L}, \tau/\mathcal{L})$ is $G/\mathcal{L}$ compact modulo $Gg\beta^*$-compact.

Proof: Let $\{\mathcal{H}_\alpha \cap \mathcal{L}: \alpha \in \Lambda\}$ be a $\tau/\mathcal{L}$ -open cover of $\mathcal{L}$, where $\mathcal{H}_\alpha \in \tau$ for each $\alpha \in \Lambda$. Now $\{\mathcal{H}_\alpha: \alpha \in \Lambda\}$ is a $Gg\beta^*$-open cover of $\mathcal{L}$ of $\mathbb{X}$. Since $\mathcal{L}$ is compact modulo $Gg\beta^*$-compact relative to $\mathbb{X}$, then there exists a finite subset Λ_0 of Λ such that $\mathcal{L} - \cup\{\mathcal{H}_\alpha: \alpha \in \Lambda_0\} \notin G$. Thus $\mathcal{L} \cap [\mathcal{L} - \cup\{\mathcal{H}_\alpha: \alpha \in \Lambda_0\}] \notin \mathcal{L} \cap G$. Since $\mathcal{L} - \cup\{\mathcal{H}_\alpha \cap \mathcal{L}: \alpha \in \Lambda_0\} = \mathcal{L} \cap [\mathcal{L} - \cup\{\mathcal{H}_\alpha: \alpha \in \Lambda_0\}] \notin \mathcal{L} \cap G$, then there exists a finite subset Λ_0 of Λ such that $\mathcal{L} - \cup\{\mathcal{H}_\alpha \cap \mathcal{L}: \alpha \in \Lambda_0\} \notin G/\mathcal{L}$. Thus $(\mathcal{L}, \tau/\mathcal{L})$ is $G/\mathcal{L}$ compact modulo $Gg\beta^*$-compact.

4.10 Definition: A subset $\mathcal{L}$ of a Grill Topological Space ($\mathbb{X}$, τ, G) is called countably - $Gg\beta^*$- compact space if every countable $Gg\beta^*$-open cover $\{\mathcal{H}_n: n \in \mathrm{N}\}$ of $\mathcal{L}$, there exists a finite subset $\{\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_k\}$ of N such that $\mathcal{L} - \cup\{\mathcal{H}_i: i = 1,2, \dots , k\} \notin G$.

4.11 Definition: A Grill Topological Space ($\mathbb{X}$, τ, G) is said to be countably - $Gg\beta^*$-compact space if it is countably -$Gg\beta^*$- compact space as a subset.

4.12 Theorem: Every compatible $Gg\beta^*$-compact space is a countably -$Gg\beta^*$-compact space.

Proof: Let ($\mathbb{X}$, τ, G) be a compatible $Gg\beta^*$-compact space. Let $\{\mathcal{H}_n: n \in \mathrm{N}\}$ be a countable $Gg\beta^*$-open cover of $\mathbb{X}$. Since $\mathbb{X}$ is a compatible $Gg\beta^*$-compact space, there exists a finite subset $\{\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_k\}$ of N such that $\mathbb{X} - \cup\{\mathcal{H}_i: i = 1,2, \dots , k\} \notin G$. This implies that ($\mathbb{X}$, τ, G) is a countably - $Gg\beta^*$-compact space.

Conclusion

We conclude that the characterizations of $Gg\beta^*$-compact space, compatible-$Gg\beta^*$- compact space and countably - $Gg\beta^*$-compact space are introduced and investigated in Grill Topological Spaces and some of the equivalent conditions are discussed. Particularly, we examined $Gg\beta^*$-compactness under different types of mappings such as $Gg\beta^*$-continuous, $Gg\beta^*$-irresolute and strongly- $Gg\beta^*$-continuous. Furthermore, these compactness notions can be extended to other generalized topological structures such as bitopological spaces, minimal structures and fuzzy Grill topological spaces.

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