



A Temperature - Structured PDE Model for Population Dynamics

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Abstract

This paper proposes a population model structured by temperature, in which thermal variation is incorporated directly as a continuous internal variable rather than treated as an external input. The resulting formulation leads to a nonlinear parabolic partial differential equation that includes diffusion along the temperature axis to represent environmental variability and redistribution effects. Population growth and decay are governed by temperature-dependent nonlinear terms of logistic type. The analytical study is conducted in an infinite-dimensional functional setting. The construction of solutions is achieved by approximating the system within finite-dimensional subspaces and then passing to the limit using suitable compactness properties together with uniform estimates. In addition, uniqueness is obtained by exploiting continuity conditions imposed on the nonlinear reaction component and applying integral inequality techniques. These results collectively guarantee the well-posedness of the model. From a biological standpoint, the formulation captures how populations adjust their distribution across heterogeneous thermal environments, offering a more realistic description than classical ordinary differential or stage-based approaches. The framework is particularly relevant in situations where temperature strongly regulates biological processes, such as in ectothermic species and vector-borne disease dynamics under climate variability. Overall, this study provides a mathematically consistent foundation for temperature-structured population models and supports further theoretical development as well as numerical exploration of environmentally structured systems.

Keywords: Temperature-structured model, non-linear parabolic partial differential equations, weak solution, well-posedness, Galerkin approximation

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1. Introduction

Temperature is one of the principal environmental factors governing the dynamics of ectothermic populations because physiological processes are directly regulated by external thermal conditions rather than internal metabolic control [1, 2]. Consequently, demographic characteristics such as development, survival, fecundity, and mortality vary systematically with temperature and often exhibit nonlinear responses with welldefined thermal optima and critical tolerance limits [1, 3]. These thermal responses determine individual fitness and, when aggregated across populations, strongly influence population persistence, geographic distribution, and ecological interactions. For mosquito populations, temperature is particularly important because it simultaneously affects immature development, adult longevity, reproductive capacity, biting frequency, and the rates of pathogen replication and transmission [4–7]. The resulting dependence of both vector and pathogen life-history traits on environmental temperature has established thermal variability as a central component of mathematical models for vector ecology and vector-borne disease dynamics. Accordingly, the development of mathematical frameworks capable of explicitly representing population responses across continuously varying thermal environments has become an important objective in theoretical ecology and mathematical biology. Over the past several decades, numerous approaches have been proposed to describe how environmental temperature shapes population dynamics. These approaches differ in their underlying assumptions but collectively provide quantitative descriptions of thermally regulated population processes. Early studies employed empirical relationships, including degree-day models for insect development [8], thermal performance curves such as the Bri`ere model [9], and mechanistic enzyme-kinetic formulations based on the Sharpe–Schoolfield equation [10], to quantify the influence of temperature on biological rates. At the population level, these thermal response functions have frequently been incorporated into systems of ordinary differential equations, where vital processes such as recruitment, maturation, reproduction, and mortality are represented by temperature-dependent coefficients [11–13]. In parallel, reaction– diffusion equations have provided a suitable framework for investigating dispersal, spatial spread, and habitat colonisation while accounting for environmental variability through temperature-dependent growth and diffusion parameters [14–18]. Collectively, these modelling approaches have significantly advanced the quantitative understanding of ecological and epidemiological processes under changing thermal conditions and have become indispensable tools for studying climate-sensitive populations. Classical ordinary differential equation and reaction–

diffusion models generally describe population dynamics through variables that depend on time or physical space, while environmental temperature is usually introduced indirectly through temperature-dependent population parameters. Such formulations commonly assume that individuals within the same spatial region experience comparable thermal conditions at a given time. Although this assumption may be suitable for relatively uniform habitats, it becomes restrictive in environments characterised by fine-scale thermal variation arising from microclimatic differences, behavioural responses, or temporal fluctuations [6, 19, 20]. Studies on mosquitoes have shown that key biological processes, including development, survival, reproduction, biting behaviour, vector competence, and pathogen growth, are influenced not only by average temperature but also by the variability and cumulative thermal exposure experienced throughout the life cycle [4, 5, 7]. These findings motivate the development of models in which temperature is incorporated directly into the population state rather than being considered solely as an external environmental driver. Structured population theory extends classical population models by introducing continuous variables that describe biologically relevant characteristics of individuals. Such variables may represent age, body size, developmental stage, physiological condition, or other traits that change over an individual's lifetime [21–24]. This approach has resulted in a broad class of partial differential equation models for age-structured, size-structured, and physiologically structured populations, providing a tool for analysing population dynamics while accounting for individual variation [23– 26]. Under this perspective, a temperature-structured formulation therefore provides a framework for describing populations whose individuals occupy different thermal states and experience distinct physiological responses. Existing developments in physiologically structured population theory have largely concentrated on classical structuring variables, including age, body size, developmental stage, and maturation level. In contrast, temperature has generally been incorporated indirectly by modifying biological rates such as growth, mortality, reproduction, or movement, rather than being considered as a variable that characterises the population itself. As a result, theoretical investigations of partial differential equation models describing populations distributed across a range of thermal states are still in an early stage. Several fundamental analytical issues associated with nonlinear temperature structured reaction–diffusion systems, including well-posedness, uniqueness, long-term dynamics, and stability of solutions, remain insufficiently understood compared with established age- and size-structured models. Addressing these mathematical challenges is essential for broadening the scope of structured population theory and developing models capable of representing populations exposed to spatially and temporally varying thermal environments. Such a framework can provide a basis for analysing ecological processes driven by temperature

variability and for understanding population-level consequences of changing thermal conditions. Motivated by these observations, we propose a nonlinear temperature-structured reaction–diffusion model in which the independent variable represents environmental temperature rather than physical space. The novelty of the proposed work lies in several aspects. First, unlike existing temperature-dependent ordinary differential equation and reaction–diffusion models, temperature is incorporated as an intrinsic state variable rather than an externally prescribed forcing parameter. Second, the model unifies continuous redistribution in thermal space with nonlinear temperature dependent logistic growth and mortality within a single partial differential equation. Third, the formulation is sufficiently general to admit functional analytic treatment based on semigroup theory and variational methods, leading to well-posedness results for weak and mild solutions. Finally, the proposed framework provides a mathematical basis for investigating population persistence, thermal adaptation, and climate-driven ecological dynamics, with potential applications to mosquito populations and other temperature-sensitive biological systems.

2. Formulation of the Model

We consider a temperature-structured population governed by the nonlinear reaction–diffusion equation

$$\frac{\partial N(T,t)}{\partial t} = D_T \frac{\partial^2 N(T,t)}{\partial T^2} + R(T)N(T,t) \left(1 - \frac{N(T,t)}{K(T)}\right) - \mu(T)N(T,t) \quad (1)$$

where $N(T, t)$ denotes the population density at temperature $T \in \Omega := (T_{\min}, T_{\max})$ and time $t > 0$. The constant $D_T > 0$ represents the diffusion coefficient in temperature space, modeling redistribution across temperature classes. The functions $R(T)$, $K(T)$, and $\mu(T)$ denote the temperature-dependent intrinsic growth rate, carrying capacity, and mortality rate, respectively. We assume throughout that $R, \mu, K \in L^\infty(\Omega)$, and $K(T) \geq K_0 > 0$ a.e. in Ω . The model is supplemented with homogeneous Neumann boundary conditions

$$\frac{\partial N}{\partial T}(T_{\min}, t) = \frac{\partial N}{\partial T}(T_{\max}, t) = 0, t > 0, \quad (2)$$

which ensure zero flux at the boundary of the temperature domain. To facilitate analysis, we reformulate (1) as an abstract Cauchy problem in the Hilbert space $X := L^2(\Omega)$. Define the linear operator $Au := D_T \frac{\partial^2 u}{\partial T^2}$ with domain $D(A) = \{u \in H^2(\Omega) : u'(T_{\min}) = u'(T_{\max}) = 0\}$. Also, define the nonlinear operator $\mathcal{F} : X \rightarrow X$ by $\mathcal{F}(u)(T) = R(T) u(T) \left(1 - \frac{u(T)}{K(T)}\right) - \mu(T) u(T)$. Then (1) can be written as the abstract Cauchy problem

$$\frac{dN}{dt} = \mathcal{A}N + \mathcal{F}(N) \quad N(0) = N_0 \in L^2(\Omega) \quad (3)$$

2.1 Biological interpretation of the model

Diffusion in Temperature Space: The term

$$D_T \frac{\partial^2 N}{\partial T^2} \quad (4)$$

represents diffusion in temperature space. The diffusion operator is interpreted analogously to Fokker-Planck equations, where diffusion acts on a continuous state variable rather than physical space. In the present model, the state variable is environmental temperature, and the diffusion term represents redistribution across neighbouring thermal conditions. [23, 26]

Temperature-Dependent Population Growth:

It is represented by the nonlinear logistic term

$$R(T)N = \left(1 - \frac{N}{K(T)}\right)$$

where the intrinsic growth rate $R(T)$ and the carrying capacity $K(T)$ are allowed to vary with environmental temperature. Experimental and theoretical studies have shown that intrinsic growth and developmental rates, are strongly influenced by temperature and frequently exhibit nonlinear thermal responses [1, 11, 27]. Allowing the carrying capacity to depend on temperature provides additional flexibility to account for changes in environmental suitability and resource availability across thermal conditions. The logistic component represents density-dependent regulation, ensuring that population growth decreases as the population density approaches the local carrying capacity [14, 22].

Temperature-Dependent Mortality:

The mortality component $-\mu(T)N$ represents temperature-dependent mortality, where $\mu(T)$ denotes the per capita death rate associated with temperature level T . In the proposed model, the mortality rate is allowed to vary continuously with temperature, thereby reflecting the influence of thermal conditions on individual survival. Such temperature-dependent mortality has been widely documented in ecological and epidemiological studies of ectothermic organisms and plays an important role in determining long-term population persistence and extinction [1, 4]. The present model is formulated within the general framework of physiologically structured population equations, in which the population density evolves with respect to a continuous state variable and time [23–26]. Figure 2 illustrates the structure of the model.

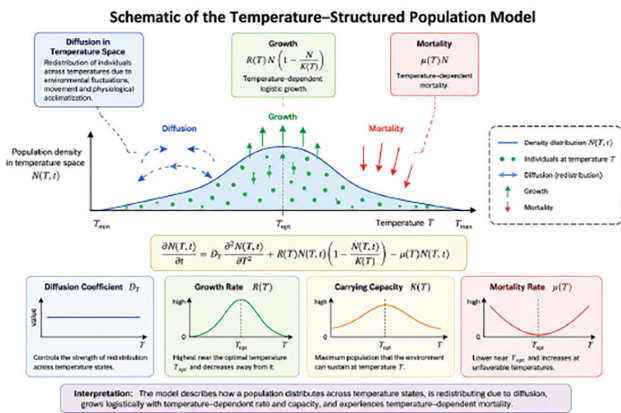


Figure 1: Schematic representation of the temperature-structured population model. Individuals are distributed across temperature states, with diffusion representing redistribution, while growth and mortality depend on temperature.

2.2. Mathematical Significance of the model

The perspectives in [25, 28] captures how temperature-dependent characteristics of individuals collectively influence the evolution of the entire population. The problem is infinite-dimensional because the unknown is a function of a continuous variable (temperature), so the solution lies in a function space like $L^2(\Omega)$, which has infinitely many degrees of freedom and hence it cannot directly solve in infinite dimensions. So we approximate using finite basis (dimension = m) and then solve the finite problem followed by taking $m \rightarrow \infty$. The reformulation in (3) interprets the solution as a trajectory in the phase space $L^2(\Omega)$, that is, $N(t) \equiv N(\cdot, t) \in L^2(\Omega)$, $t \geq 0$. The diffusion operator in (1) is incorporated into the linear operator A , allowing the system to be studied using semigroup methods for abstract evolution equations in infinite-dimensional spaces.

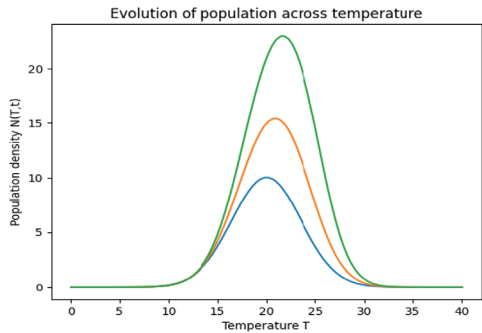


Figure 2: Evolution of the population density $N(T,t)$ across temperature at different time points. The distribution shifts and concentrates near an optimal temperature due to the interaction of diffusion, growth, and mortality.

2.3. Functional Spaces and Notation

This section presents the functional spaces and notation required for establishing the existence theorem.

Initial condition: $N_0 \in L^2(\Omega)$.

$L^2(\Omega) = \{f : \Omega \rightarrow \mathbb{R} \mid \int_{\Omega} |f(T)|^2 dT < \infty\}$. This means that the initial population distribution $N_0(T)$ is square integrable. In particular, the total population energy is finite.

Coefficients: $R(T), K(T), \mu(T) \in L^\infty(\Omega)$.

The space $L^\infty(\Omega) = \{f : \Omega \rightarrow \mathbb{R} \mid \exists C > 0 \text{ such that } |f(T)| \leq C \text{ a.e.}\}$. This assumption ensures that the growth rate, carrying capacity, and mortality rate remain bounded, which is biologically realistic.

Solution space: $N \in L^2(0, t_f; H^1(\Omega))$.

The Sobolev space $H^1(\Omega) = \{f \in L^2(\Omega) \mid \frac{\partial f}{\partial T} \in L^2(\Omega)\}$. The functions in $H^1(\Omega)$ are square integrable together with their first partial derivative. The space $L^2(0, t_f; H^1(\Omega))$ consists of functions such that $\int_0^{t_f} \|N(t)\|_{H^1(\Omega)}^2 dt < \infty$ where the symbol t_f denotes the final time (or terminal time) of the time interval over which the solution is considered. Thus, the solution is defined on the space-time domain $\Omega \times (0, t_f)$.

Time derivative: $\partial_t N \in L^2(0, t_f; H^{-1}(\Omega))$

The space $H^{-1}(\Omega) = (H^1(\Omega))'$. The condition $\partial_t N \in L^2(0, t_f; H^{-1}(\Omega))$ means that the time derivative of N exists in a weak (distributional) equation:

$$\frac{\partial N}{\partial t} = D_T \frac{\partial^2 N}{\partial T^2} + F(N), \quad (5)$$

where $F(N) = R(T)N \left(1 - \frac{N}{K(T)}\right) - \mu(T)N$.

Let $v \in H^1(\Omega)$ be a test function. Multiplying (5) by v and integrating over Ω , we obtain:

$$\int_{\Omega} \frac{\partial N}{\partial t} v \, dT = D_T \int_{\Omega} \frac{\partial^2 N}{\partial T^2} v \, dT + \int_{\Omega} F(N) v \, dT. \quad (6)$$

To avoid second-order derivatives, we apply integration by parts:

$$\int_{\Omega} \frac{\partial^2 N}{\partial T^2} v \, dT = - \int_{\Omega} \frac{\partial N}{\partial T} \frac{\partial v}{\partial T} \, dT$$

where the boundary term vanishes due to homogeneous Neumann boundary conditions. Thus, the weak formulation becomes:

$$\int_{\Omega} \frac{\partial N}{\partial T} v \, dT + D_T \int_{\Omega} \frac{\partial N}{\partial T} \frac{\partial v}{\partial T} \, dT = \int_{\Omega} F(N) v \, dT. \quad (7)$$

The term

$$\int_{\Omega} \frac{\partial N}{\partial T} v \, dT$$

requires the existence of the time derivative $\partial_t N$. However, in many cases, $\partial_t N$ may not exist as a classical function. To overcome this difficulty, we interpret the time derivative in a weak sense using duality. Specifically, we write:

$$\int_{\Omega} \frac{\partial N}{\partial T} v \, dT = \langle \partial_t N, v \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the dual pairing between $H^{-1}(\Omega)$ and $H^1(\Omega)$.

The space $H^{-1}(\Omega)$ is defined as the dual space of $H^1(\Omega)$:

Thus, $\partial_t N \in H^{-1}(\Omega)$ means that the time derivative acts as a bounded linear functional on $H^1(\Omega)$. The weak formulation of the problem can therefore be written as:

$$\langle \partial_t N, v \rangle + D_T \int_{\Omega} \frac{\partial N}{\partial T} \frac{\partial v}{\partial T} \, dT = \int_{\Omega} F(N) v \, dT, \quad \forall v \in H^1(\Omega). \quad (8)$$

This formulation shows that even if $\partial_t N$ does not exist pointwise, it can still be meaningfully interpreted as an element of $H^{-1}(\Omega)$.

This allows the problem to be well-defined in a weak (variational) sense. This allows us to treat solutions that may not be classically differentiable in time, but still satisfy the equation when tested against smooth functions. The above conditions imply that the solution $N(T, t)$ is square integrable in space and time, has square integrable spatial derivatives and admits a weak time derivative (which is sufficient to interpret the PDE in the weak variational sense).

3. Analysis of the Linear and Non-linear Operator

Proposition 1 *The operator A is linear on $D(A)$.*

Proof Let $u, v \in D(A)$ and $\alpha, \beta \in \mathbb{R}$. Then,

$$A(\alpha u + \beta v) = D_T(\alpha u + \beta v)'' = \alpha D_T u'' + \beta D_T v'' = \alpha A u + \beta A v.$$

Hence, A is linear.

Proposition 2 *The operator A is unbounded in $L^2(\Omega)$.*

Proof Let

$$u_n(T) = \cos\left(\frac{n\pi(T-T_{min})}{L}\right), L = T_{max} - T_{min}.$$

Then $\{u_n\} \subset D(A)$ and $\|u_n\|_{L^2(\Omega)}$ is uniformly bounded. Moreover,

$$Au_n = -D_T \left(\frac{n\pi}{L}\right)^2 u_n.$$

Hence,

$$\|Au_n\|_{L^2(\Omega)} = D_T \left(\frac{n\pi}{L}\right)^2 \|u_n\|_{L^2(\Omega)},$$

which implies

$$\frac{\|Au_n\|_{L^2(\Omega)}}{\|u_n\|_{L^2(\Omega)}} \rightarrow \infty \text{ as } n \rightarrow \infty$$

Therefore, A is unbounded.

Proposition 3 The domain $D(A)$ is dense in $L^2(\Omega)$.

Proof The operator A corresponds to the Neumann Laplacian on Ω . It is well known that its eigenfunctions

$$u_n(T) = \cos\left(\frac{n\pi(T-T_{min})}{L}\right), n \geq 0,$$

form a complete orthogonal basis of $L^2(\Omega)$.

Since finite linear combinations of these eigenfunctions belong to $D(A)$, and these combinations are dense in $L^2(\Omega)$, it follows that $D(A)$ is dense in $L^2(\Omega)$.

Proposition 4 The operator A with homogeneous Neumann boundary conditions is self-adjoint on $L^2(\Omega)$.

Proof The operator $A = D_T \frac{d^2}{dT^2}$ with domain

$$D(A) = \{u \in H^2(\Omega): u'(T_{min}) = u'(T_{max}) = 0\}$$

is a positive scalar multiple of the Neumann Laplacian. It is a classical result that the Neumann Laplacian on a bounded interval defines a densely defined, closed, self-adjoint operator in $L^2(\Omega)$. Since multiplication by the positive constant D_T preserves self-adjointness and does not change the domain, it follows that A is self-adjoint in $L^2(\Omega)$.

Proposition 5 The operator A is dissipative on $L^2(\Omega)$.

Proof Let $u \in D(A)$. Using integration by parts and the homogeneous Neumann boundary conditions, we obtain

$$\langle Au, u \rangle_{L^2(\Omega)} = D_T \int_{\Omega} u''(T) u(T) dt = -D_T \int_{\Omega} |u'(T)|^2 dT \leq 0.$$

Hence, A is dissipative.

Proposition 6 *The operator A with homogeneous Neumann boundary conditions is maximal dissipative on $L^2(\Omega)$.*

Proof We already know that A is dissipative. It remains to show that for some $\lambda > 0$ and every $f \in L^2(\Omega)$, there exists $u \in D(A)$ such that $(\lambda I - A)u = f$. Consider the weak formulation, find $u \in H^1(\Omega)$ such that

$$\lambda \int_{\Omega} uv dT + D_T \int_{\Omega} u'v' dT = \int_{\Omega} fvdT, \forall v \in H^1(\Omega).$$

Define the bilinear form

$$a(u, v) = \lambda \int_{\Omega} uv dT + D_T \int_{\Omega} u'v' dT.$$

Then $a(\cdot, \cdot)$ is continuous on $H^1(\Omega)$ and satisfies $a(u, v) \geq c \|u\|_{H^1(\Omega)}^2$ for some constant $c > 0$. Thus, the linear functional $\int_{\Omega} fvdT$ is continuous on $H^1(\Omega)$.

By the Lax-Milgram theorem, there exists a unique $u \in H^1(\Omega)$ solving the weak problem. Standard elliptic regularity for the one-dimensional Neumann problem implies $u \in H^2(\Omega)$ and $\partial_T u = 0$ on $\partial\Omega$. Hence $u \in D(A)$ and $(\lambda I - A)u = f$. Therefore, $\text{Range } \lambda I - A = L^2(\Omega)$ and A is maximal dissipative.

Proposition 7 *The operator $F: L^2(\Omega) \rightarrow L^2(\Omega)$ defined by*

$$\mathcal{F}(u)(T) = (R(T) - \mu(T))u(T) - \frac{R(T)}{K(T)}u(T)^2$$

is nonlinear.

Proof The presence of the quadratic term u^2 implies that

$$F(\alpha u + \beta v) \neq \alpha F(u) + \beta F(v)$$

In general, hence F is nonlinear.

Proposition 8 *Assume $R, \mu \in L^\infty(\Omega)$ and $K \in L^\infty(\Omega)$ with $K(T) \geq k_0 > 0$ a.e. in Ω . Then $F: L^2(\Omega) \rightarrow L^2(\Omega)$ is locally Lipschitz continuous.*

Proof Let $u, v \in L^2(\Omega)$. Then

$$F(u) - F(v) = (R - \mu)(u - v) - \frac{R}{K}(u^2 - v^2).$$

Using $u^2 - v^2 = (u-v)(u+v)$, we obtain

$$\|F(u) - F(v)\|_{L^2} \leq \|R - \mu\|_{L^\infty} \|u - v\|_{L^2} + \frac{\|R\|}{\|K\|_{L^\infty}} \| (u - v)(u + v) \|_{L^2}$$

For the nonlinear term, by Holder's inequality,

$$\| (u - v)(u + v) \|_{L^2} \leq \|u - v\|_{L^4} \|u + v\|_{L^4}.$$

Standard interpolation on bounded subsets of $L^2(\Omega)$, yielding

$$\|F(u) - F(v)\|_{L^2} \leq C(1 + \|u\|_{L^2} + \|v\|_{L^2}) \|u - v\|_{L^2}.$$

Thus F is locally Lipschitz on $L^2(\Omega)$.

4. Qualitative Properties

Theorem 9 (Positivity, Boundedness and Invariant Region) *Assume*

$$D_T > 0, R, \mu, K \in L^\infty(\Omega), K(T) \geq K_0 > 0 \text{ a. e. in } \Omega$$

Let

be a weak solution of (1) with initial datum satisfying

$$0 \leq N_0(T) \leq K(T) \text{ a. e. in } \Omega.$$

Define the closed context set

$$\mathcal{R} := \{u \in L^2(\Omega; 0 \leq u(T) \leq K(T) \text{ a. e. in } \Omega)\}$$

Then the following hold:

1. (Positivity) $N(T, t) \geq 0$ a.e. in $\Omega \times (0, t_f)$.
2. (Upper bound) $N(T, t) \leq K(T)$ a.e. in $\Omega \times (0, t_f)$.
3. (Invariant region) $N(t) \in \mathcal{R}$ for all $t \in [0, t_f]$.
4. (Uniform bound)

$$\|N(t)\|_{L^\infty(\Omega)} \leq \|K\|_{L^\infty(\Omega)} \quad \forall t \in [0, t_f]$$

Proof We prove invariance of the convex set $\mathcal{R}$ by establishing separately the lower and upper constraints.

Positivity. Let $N^- = \max\{-N, 0\}$ denote the negative part of N . Testing the equation with N^- and using the Neumann boundary conditions yields the energy identity.

$$\frac{1}{2} \frac{d}{dt} \|N^-(t)\|_{L^2(\Omega)}^2 = -D_T \|\partial_T N^-\|_{L^2(\Omega)}^2 + \int_\Omega F(N) N^- dT.$$

where

$$F(N) = R(T)N \left(1 - \frac{N}{K(T)}\right) - \mu(T)N$$

On $\{N < 0\}$ we have $N = -N^-$, hence $F(N)N^- \leq 0$. Therefore,

$$\frac{d}{dt} \|N^-(t)\|_{L^2(\Omega)}^2 \leq 0$$

Since $N_0 \geq 0$, we obtain $N^- \equiv 0$, proving positivity.

Upper bound. Define the positive part of the deviation from the carrying capacity:

$$W := (N - K)^+.$$

Testing the equation with W and using that K is time-independent gives

$$\frac{1}{2} \frac{d}{dt} \|W(t)\|_{L^2(\Omega)}^2 = -D_T \|\partial_T W\|_{L^2(\Omega)}^2 + \int_{\Omega} F(N)W dT.$$

On $\{N > K\}$, we have $N > K(T)$ and thus

$$1 - \frac{N}{K(T)} \leq 0$$

Since $N \geq 0$, it follows that $F(N) \leq 0$, hence

$$F(N)W \leq 0.$$

Therefore,

$$\frac{d}{dt} \|W(t)\|_{L^2(\Omega)}^2 \leq 0.$$

With $W(0) = 0$, we conclude $W \equiv 0$, i.e. $N \leq K$.

Invariant region: Combining Steps 1 and 2 yields

$$0 \leq N(T, t) \leq K(T) \quad \text{a.e. in } \Omega \times (0, t_f),$$

which is equivalent to

$$N(t) \in \mathcal{R} \quad \forall t \in [0, t_f]$$

L^∞ bound. Taking essential supremum gives

$$\|N(t)\|_{L^\infty(\Omega)} \leq \|K\|_{L^\infty(\Omega)}.$$

The proof is complete.

5. Well-Posedness of the Model

In this section, we establish the existence and uniqueness of weak solutions to the temperature-structured population model.

5.1 Existence of Weak Solutions

Theorem 10 (Existence of Weak Solutions) *Assume that:*

- $N_0 \in L^2(\Omega)$,
- $R(T), K(T), \mu(T) \in L^\infty(\Omega)$ with $K(T) \geq K_0 > 0$.

Then there exists a weak solution

$$N \in L^2(0, t_f; H^1(\Omega)), \partial_t N \in L^2(0, t_f; H^{-1}(\Omega))$$

satisfying (5) to (7).

Proof: Step 1: Galerin Approximation

Let $\{\phi_k\}_{k=1}^\infty \subset H^1(\Omega)$ be an orthonormal basis of $L^2(\Omega)$, consisting of eigenfunctions of the Laplacian with homogeneous Neumann boundary conditions.

$$-\Delta \phi_k = \lambda_k \phi_k, \quad \frac{\partial \phi_k}{\partial \nu} = 0 \text{ on } \partial\Omega.$$

Define the finite-dimensional space

$$V_m = \text{span}\{\phi_1, \phi_2, \dots, \phi_m\}$$

We seek an approximate solution of the form

$$N_m(T, t) = \sum_{i=1}^m g_i^{(m)}(t) \phi_i(T)$$

where $g_i^{(m)}(t)$ are unknown coefficients.

We denote the L^2 -inner product by

$$(u, v) := \int_{\omega} u(T)v(T)dT.$$

Step 2: Galerkin System

For each $j=1, \dots, m$, we require:

$$\left(\frac{\partial N_m}{\partial t}, \phi_j\right) + D_T(\nabla N_m, \nabla \phi_j) = \left(R(T)N_m\left(1 - \frac{N_m}{K(T)}\right), \phi_j\right) - (\mu(T)N_m, \phi_j). \quad (9)$$

With initial condition:

$$N_m(0) = N_{0m}$$

Where N_{0m} is the L^2 - projection of N_0 onto V_m .

Substituting the expansion of N_m , we obtain a system of nonlinear ordinary differential equations for the coefficients:

$$g_j^{(m)}(t) + D_T \sum_{i=1}^m g_i^{(m)}(t)(\nabla \phi_i, \nabla \phi_j) = F_j(g_1^{(m)}, \dots, g_m^{(m)}),$$

for $j = 1, \dots, m$ where F_j represents the nonlinear reaction terms.

Step 3: Local Existence

Since the nonlinear terms are locally Lipschitz with respect to the coefficients $g_i^{(m)}$, standard ODE theory ensures the existence of a local solution.

$$g_i^{(m)}(t) \in C^1([0, t_m])$$

Step 4: A Priori Estimates

Take $\varphi = N_m$, in (9).

$$\left(\frac{\partial N_m}{\partial t}, N_m\right) + D_T \|\nabla N_m\|_{L^2}^2 = \int_{\Omega} R(T)N_m^2 \left(1 - \frac{N_m}{K(T)}\right) dT - \int_{\Omega} \mu(T)N_m^2 dT \quad (10)$$

Observe that:

$$\left(\frac{\partial N_m}{\partial t}, N_m\right) = \frac{1}{2} \frac{d}{dt} \|N_m\|_{L^2}^2$$

Using boundedness of R and μ , we estimate:

$$\int_{\Omega} R(T)N_m^2 \leq \|R\|_{\infty} \|N_m\|_{L^2}^2, \int_{\Omega} \mu(T)N_m^2 \leq \|\mu\|_{\infty} \|N_m\|_{L^2}^2 \quad (11)$$

Applying Gronwall's inequality yields:

$$\| N_m(t) \|_{L^2}^2 dt \leq C.$$

Hence,

N_m is bounded in $L^2(0, t_f; H^1(\Omega))$.

Step 5: Estimate of Time Derivative

For any $\varphi \in H^1(\Omega)$ from (9):

$$(\partial_t N_m, \varphi) = -D_T(\nabla N_m, \nabla \varphi) + \left(RN_m \left(1 - \frac{N_m}{K} \right), \varphi \right) - (\mu N_m, \varphi)$$

Using Hölder's inequality and the Sobolev embedding $H^1(\Omega) \rightarrow L^4(\Omega)$ (valid for spatial dimension $d \leq 3$), we obtain:

$$\| RN_m \left(1 - \frac{N_m}{K} \right) \|_{L^2} \leq C (\| N_m \|_{L^2} + \| N_m \|_{L^4}^2) \leq C (\| N_m \|_{L^2} + \| N_m \|_{H^1}^2).$$

Thus,

$$|(\partial_t N_m, \varphi)| \leq C (\| N_m \|_{H^1} + \| N_m \|_{L^2}^2) \| \varphi \|_{H^1}.$$

Hence,

$$\| \partial_t N_m \|_{H^{-1}} \leq C (\| N_m \|_{H^1} + \| N_m \|_{L^2}^2),$$

And therefore,

$\partial_t N_m$ is bounded in $L^2(0, t_f; H^{-1}(\Omega))$.

Step 6: Compactness

We have:

N_m bounded in $L^2(0, t_f; H^1(\Omega))$, $\partial_t N_m$ bounded in $L^2(0, t_f; H^{-1}(\Omega))$.

Since the embeddings

$$H^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$$

Hold, the Aubin-Lions lemma implies:

$$N_m \rightarrow N, L^2(0, t_f; L^2(\Omega))$$

Step 7: Passage to the Limit

Strong convergence implies:

$$N_m \rightarrow N, N_m^2 \rightarrow N^2 \text{ in } L^1(\Omega \times (0, t_f)).$$

Thus,

$$R(T)N_m \left(1 - \frac{N_m}{K(T)}\right) \rightarrow R(T)N \left(1 - \frac{N}{K(T)}\right)$$

Passing to the limit in (9), we obtain the weak foundation.

Step 8: Initial Condition

Since

$$N_{0m} \rightarrow N_0 \text{ in } L^2(\Omega),$$

and

$$Nm \rightarrow N \text{ in } C([0, t_f]; H^{-1}(\Omega)),$$

We conclude:

$$N(0) = N_0.$$

Conclusion

There exists a function

$$N \in L^2(0, t_f; H^1(\Omega)), \partial_t N \in L^2(0, t_f; H^{-1}(\Omega))$$

which satisfies the weak formulation and the initial condition.

5.2 Uniqueness of weak solution

Theorem 11 Let $N_1, N_2 \in L^2(0, t_f; H^1(\Omega))$ with $\partial_t N_1, \partial_t N_2 \in L^2(0, t_f; H^{-1}(\Omega))$ be two weak solutions of problem (1) corresponding to the same initial data $N_0 \in L^2(\Omega)$.

$$F(N) = R(T)N \left(1 - \frac{N}{K(T)}\right) - \mu(T)N$$

is Lipschitz continuous in N , i.e., there exists a constant $C > 0$ such that

$$|F(N_1) - F(N_2)| \leq C |N_1 - N_2|.$$

Then the weak solution is unique, i.e.,

$N_1 = N_2$ almost everywhere in $\Omega \times (0, t_f)$.

Proof Let $W = N_1 - N_2$. Subtracting the weak formulations satisfied by N_1 and N_2 , we obtain:

$$\frac{\partial W}{\partial t} = D_T \frac{\partial^2 W}{\partial T^2} + (F(N_1) - F(N_2)) \quad (12)$$

We test (12) with W and integrate over Ω :

$$\int_{\Omega} \frac{\partial W}{\partial t} W dT = D_T \int_{\Omega} \frac{\partial^2 W}{\partial T^2} W dT + \int_{\Omega} (F(N_1) - F(N_2)) W dT \quad (13)$$

Time derivative term

$$\int_{\Omega} \frac{\partial W}{\partial t} W dT = \frac{1}{2} \frac{d}{dt} \|W\|_{L^2(\Omega)}^2$$

Using integration by parts and the homogeneous Neumann boundary conditions the diffusion term,

$$D_T \int_{\Omega} \frac{\partial^2 W}{\partial T^2} W dT = -D_T \int_{\Omega} |\partial_T W|^2 dT \leq 0.$$

By the Lipschitz continuity of F ,

$$\left| \int_{\Omega} (F(N_1) - F(N_2)) W dT \right| \leq C \int_{\Omega} |W|^2 dT = C \|W\|_{L^2(\Omega)}^2$$

Combining the above estimates, we obtain:

$$\frac{1}{2} \frac{d}{dt} \|W\|_{L^2(\Omega)}^2 \leq C \|W\|_{L^2(\Omega)}^2$$

Applying Grönwall's inequality yields:

$$\|W(t)\|_{L^2(\Omega)}^2 \leq \|W(0)\|_{L^2(\Omega)}^2 e^{Ct}$$

Since $N_1(0) = N_2(0) = N_0$, we have $W(0) = 0$, and therefore

$$\|W(t)\|_{L^2(\Omega)}^2 = 0 \quad \forall t \in (0, t_f).$$

Hence, $W = 0$ almost everywhere, and consequently,

$N_1 = N_2$ almost everywhere in $\Omega \times (0, t_f)$.

This completes the proof.

Remark 1 The above results establish the well-posedness of the model. The Galerkin method provides a constructive existence proof, while uniqueness follows from energy estimates and Grönwall's inequality.

6 Conclusion and Further Work

This work developed and analysed a nonlinear temperature-structured population model formulated as a parabolic partial differential equation in which temperature serves as a continuous structuring variable. The model incorporates diffusion in temperature space together with temperature-dependent growth and mortality, allowing the population to redistribute across heterogeneous thermal environments. From a mathematical perspective, we established the existence of weak solutions using a Galerkin approximation, supported by a priori estimates and compactness arguments, and proved uniqueness under a Lipschitz condition on the reaction term.

These results ensure that the model is well posed and provide a basis for further analytical and numerical investigations. The present formulation suggests a number of directions for further study. On the analytical side, it would be of interest to examine the long-time behavior of solutions, in particular the existence and stability of steady states associated with the temperature-dependent coefficients. However, proving the uniqueness of steady states, local exponential stability, or the existence of a global attractor remain open within the present framework and constitute natural directions for future research. The model could also be extended by coupling the temperature variable with spatial movement, leading to a system in which individuals are structured both by location and thermal exposure. In summary, this work establishes a mathematical foundation for population dynamics by combining nonlinear parabolic PDE theory with ecological modelling.

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Author Contributions

Alphy Joseph developed the model and performed the analysis. Krishnadas Ashok contributed to the mathematical formulation and proofs. Fijy Jose P assisted with interpretation and manuscript preparation. All authors reviewed and approved the final manuscript.

Conflict of Interest

The authors declare no conflict of interest.

References

- [1] Angilletta, J. Michael J.: Thermal Adaptation: A Theoretical and Empirical Synthesis. Oxford University Press, Oxford, UK (2009). <https://doi.org/10.1093/acprof:oso/9780198570872.001.0001>
- [2] Huey, R.B., Kingsolver, J.G.: Evolution of thermal sensitivity of ectotherm performance. *Trends in Ecology & Evolution* 4(5), 131–135 (1989) [https://doi.org/10.1016/0169-5347\(89\)90211-5](https://doi.org/10.1016/0169-5347(89)90211-5)
- [3] Kingsolver, J.G.: The well-temperated biologist. *The American Naturalist* 174(6), 755–768 (2009) <https://doi.org/10.1086/648310>
- [4] Mordecai, E.A., Caldwell, J.M., Grossman, M.K., Lippi, C.A., Johnson, L.R., Neira, M., Rohr, J.R., Ryan, S.J., Savage, V.M., Shocket, M.S., Thomas, M.B.: Thermal biology of mosquito-borne disease. *Ecology Letters* 22(10), 1690–1708 (2019) <https://doi.org/10.1111/ele.13335>
- [5] Lambrechts, L., Paaijmans, K.P., Fansiri, T., et al.: Impact of daily temperature fluctuations on dengue virus transmission by aedes aegypti. *Proceedings of the National Academy of Sciences* 108(18), 7460–7465 (2011) <https://doi.org/10.1073/pnas.1101377108>
- [6] Paaijmans, K.P., Read, A.F., Thomas, M.B.: Understanding the link between malaria risk and climate. *Proceedings of the National Academy of Sciences* 107(34), 15135–15139 (2010) <https://doi.org/10.1073/pnas.1006422107>
- [7] Tesla, B., Demakovsky, L.R., Mordecai, E.A., Ryan, S.J., Bonds, M.H., Ngonghala, C.N., Brindley, M.A., Murdock, C.C.: Temperature drives Zika virus transmission: Evidence from empirical and mathematical models. *Proceedings of the Royal Society B: Biological Sciences* 285(1884), 20180795 (2018) <https://doi.org/10.1098/rspb.2018.0795>
- [8] Higley, L.G., Pedigo, L.P., Ostlie, K.R.: Degday: A program for calculating degree-days. *Environmental Entomology* 15(5), 999–1013 (1986) <https://doi.org/10.1093/ee/15.5.999>
- [9] Bri`ere, J.-F., Pracros, P., Le Roux, A.-Y., Pierre, J.-S.: A novel rate model of temperature-dependent development for arthropods. *Environmental Entomology* 28(1), 22–29 (1999) <https://doi.org/10.1093/ee/28.1.22>
- [10] Schoolfield, R.M., Sharpe, P.J.H., Magnuson, C.E.: Non-linear regression of biological temperature-dependent rate models based on absolute reaction-rate theory. *Journal of Theoretical Biology* 88(4), 719–731 (1981) [https://doi.org/10.1016/0022-5193\(81\)90246-0](https://doi.org/10.1016/0022-5193(81)90246-0)

- [11] Amarasekare, P.: Effects of temperature on consumer–resource interactions. *Journal of Animal Ecology* 84(3), 665–679 (2015) <https://doi.org/10.1111/1365-2656.12320>
- [12] Smith, D.L., Battle, K.E., Hay, S.I., Barker, C.M., Scott, T.W., McKenzie, F.E.: Ross, macdonald, and a theory for the dynamics and control of mosquito-transmitted pathogens. *PLoS Pathogens* 8(4), 1002588 (2012) <https://doi.org/10.1371/journal.ppat.1002588>
- [13] Mordecai, E.A., Cohen, J.M., Evans, M.V., Gudapati, P., Johnson, L.K., Lippi, C.A., Miazgowicz, K., Murdock, C.C., Rohr, J.R., Ryan, S.J., Savage, V.M., 18 Shocket, M.S., Ibarra, A.M., Thomas, M.B., Weikel, D.P.: Detecting the impact of temperature on transmission of Zika, dengue, and chikungunya using mechanistic models. *PLoS Neglected Tropical Diseases* 11(4), 0005568 (2017) <https://doi.org/10.1371/journal.pntd.0005568>
- [14] Murray, J.D.: *Mathematical Biology II: Spatial Models and Biomedical Applications*, 3rd edn. *Interdisciplinary Applied Mathematics*, vol. 18. Springer, New York (2003). <https://doi.org/10.1007/b98869>
- [15] Okubo, A., Levin, S.A.: *Diffusion and Ecological Problems: Modern Perspectives*, 2nd edn. Springer, New York (2001). <https://doi.org/10.1007/978-1-4757-4978-6>
- [16] Richter, O., et al.: Modelling the effect of temperature on the range expansion of species by reaction–diffusion equations. *Mathematical Biosciences* 235(2), 171–181 (2012) <https://doi.org/10.1016/j.mbs.2011.12.001>
- [17] Cosner, C.: Reaction–diffusion equations and ecological modeling. In: *Tutorials in Mathematical Biosciences IV. Lecture Notes in Mathematics*, vol. 1922, pp. 77–115. Springer, ??? (2008). https://doi.org/10.1007/978-3-540-74331-6_3
- [18] Roitberg, E., Shoshany, M.: Can spatial patterns along climatic gradients predict ecosystem responses to climate change? experimenting with reaction–diffusion simulations. *PLoS ONE* 12(4), 0174942 (2017) <https://doi.org/10.1371/journal.pone.0174942>
- [19] Reiner, R.C.J., Perkins, T.A., Barker, C.M., Niu, T., Chaves, L.F., Ellis, A.M., George, D.B., Le Menach, A., Pulliam, J.R.C., Bisanzio, D., Buckee, C.O., Chiyaka, C., Cummings, D.A.T., Garcia, A.J., Gattton, M.L., Gething, P.W., Hartley, D.M., Johnston, G., Klein, E.Y., Michael, E., Lindsay, S.W., Lloyd, A.L., Pigott, D.M., Reisen, W.K., Ruktanonchai, N.W., Singh, B., Tatem, A.J., Kitron, U., Hay, S.I., Scott, T.W., Smith, D.L.: A systematic review of mathematical models of mosquito-borne pathogen transmission. *Journal of The Royal Society Interface* 13(122), 20160921 (2016) <https://doi.org/10.1098/rsif.2016.0921>
- [20] Parham, P.E., Michael, E.: Modeling the effects of weather and climate change on malaria transmission. *Environmental Health Perspectives* 118(5), 620–626 (2010) <https://doi.org/10.1289/ehp.0901256>
- [21] Webb, G.F.: *Theory of Nonlinear Age-Dependent Population Dynamics. Monographs and Textbooks in Pure and Applied Mathematics*, vol. 89. Marcel Dekker, New York (1985). <https://doi.org/10.1201/9780203742402>

- [22] Cushing, J.M.: An Introduction to Structured Population Dynamics. SIAM, Philadelphia (1998). <https://doi.org/10.1137/1.9781611970088> [23] Perthame, B.: Transport Equations in Biology. Frontiers in Mathematics. Birkh auser Basel, Basel, Switzerland (2007). <https://doi.org/10.1007/19978-3-7643-7842-5>
- [24] Inaba, H.: Age-Structured Population Dynamics in Demography and Epidemiology. Springer, Singapore (2017). <https://doi.org/10.1007/978-981-10-0187-8> [25] Metz, J.A.J., Diekmann, O. (eds.): The Dynamics of Physiologically Structured Populations. Lecture Notes in Biomathematics, vol. 68. Springer, Berlin (1986). <https://doi.org/10.1007/978-3-662-13159-6>
- [26] Diekmann, O., et al.: Physiologically structured population models: towards a general mathematical theory. Journal of Mathematical Biology (1998)
- [27] Gillooly, J.F., Brown, J.H., West, G.B., Savage, V.M., Charnov, E.L.: Effects of size and temperature on metabolic rate. Science 293(5538), 2248–2251 (2001) <https://doi.org/10.1126/science.1061967>
- [28] Metz, J.A.J., Diekmann, O. (eds.): The Dynamics of Physiologically Structured Populations. Lecture Notes in Biomathematics, vol. 68. Springer, Berlin, Heidelberg (1986). <https://doi.org/10.1007/978-3-662-13159-6>