

Balanced Traffic Splitting in Communication Network Using Parity Sum Cordial Labeling of Graphs

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Abstract

Labeling of graph is the assignment of integers to the elements of a graph such as vertices, edges or both. This process gives an easiest way of modelling real world networks. Different types of labeling are available in the literature. Parity sum cordial labeling is a new graph labeling that we defined in this paper. Let G_1 be a (p, q) graph. Let f_v be an injective map from vertex set of G_1 to $\{1, 2, \dots, p\}$. For each edge $\{x, y\}$ assign the label $f_v(x) + f_v(y)$. f_v is named a parity sum cordial labeling (PSC-labeling), if f_v is a bijective map and $|e(0) - e(1)| \leq 1$, where $e(0)$ and $e(1)$ denote the number of edges labelled with even numbers and odd numbers respectively. A graph which follows parity sum cordial labeling is a parity sum cordial graph (PSC graph). Also we investigate the PSC-labeling behaviour of path graph, cycle graph, comb graph, complete graph and illustrated Balanced Traffic Splitting in communication network using PSC Labeling. With this routers or load balancers can direct traffic based on parity and maintain equal load on both ports. It helps to avoid server overload.

Keywords: Binary vertex labeling, Cordial labeling, parity sum cordial labeling, parity sum cordial graph, Balanced Traffic Splitting.

1. Introduction

Graph theory is used to model different types of networks in real world. It has several branches and labeling of graph is one of it. Graph labeling is becoming more interesting field because of its wide range of applications. Graph labeling played an important role in different fields. Alex Rosa [7] introduced graph labelings in 1960s.

The cordial labeling of graph was introduced by Cahit [1] in 1987. Different types of graph classes were proved as cordial graphs. R. Ponraj et al. [5] introduced Parity Combination cordial labeling of graphs. Motivated

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by this labeling, we introduce a new graph labeling called parity sum cordial labelling (PSC-labeling). In this paper, we investigate the PSC-labeling nature of different types of graph families such as path graph, cycle graph, comb graph, complete graph and illustrated Balanced Traffic Splitting in communication network using PSC Labeling. Compared to existing graph labelings, the novel features of PSC labeling include unique vertex identification, parity - based edge classification and computational simplicity. The graphs here we considered are simple, undirected and finite and the number of vertices is denoted by p and number of edges denoted by q .

2. Prerequisites

Definition 2.1 [4] A nonempty set $V_1(G_1)$ of vertices, also known as nodes, plus a set $E_1(G_1)$ of edges connecting the vertices make up a graph $G_1 = (V_1(G_1), E_1(G_1))$. The endpoints of each edge are the one or two vertices that are connected to it.

Definition 2.2 [3,7] An integer assignment to the vertices, edges, or both, subject to specific requirements, is known as a graph labeling.

Definition 2.3 [1] Let the graph be $G_1 = (V_1(G_1), E_1(G_1))$. Binary vertex labeling of G_1 is defined as a mapping $f_v: V_1(G_1) \rightarrow \{0, 1\}$, where $f_v(v)$ is the label of the vertex v of G_1 under f_v .

Definition 2.4 [1] Let the graph be $G_1 = (V_1(G_1), E_1(G_1))$. A binary vertex labeling of G_1 is defined by $f_v: V_1(G_1) \rightarrow \{0, 1\}$. The edge labeling $f_{e^*}: E_1(G_1) \rightarrow \{0, 1\}$ is given by $f_{e^*}(e^*) = |f_v(x) - f_v(y)|$ for each edge $e^* = xy$ in $E_1(G_1)$. According to Cahit "a cordial labeling is a binary vertex labeling which satisfies $|v(0) - v(1)| \leq 1$ and $|e(0) - e(1)| \leq 1$ where $v(0)$, $v(1)$ be the number of vertices of G_1 with labels zero and one respectively under f_v and let $e(0)$, $e(1)$ be the number of edges with labels zero and one respectively under f_{e^*} . A graph G_1 is called cordial if it admits a cordial labelling".

Definition 2.5 [5] Consider a graph G_1 with p vertices and q edges. Let f_v be a one-to-one function from vertex set of G_1 to $\{1, 2, \dots, p\}$. For every edge $\{x, y\}$, mark the label $x C_y$ or $y C_x$ according as $x > y$ or $y > x$. According to R. Ponraj et al. " f_v is called a parity combination cordial labeling (PCC-labeling) if f_v is a one to one function and $|e(0) - e(1)| \leq 1$ where $e(1)$ and $e(0)$ represent the number of edges marked with odd numbers and even numbers respectively".

3. PSC-labeling

Definition 3.1 Consider a graph G_1 with p vertices and q edges. Let f_v be a one-to-one and onto function from vertex set of G_1 to $\{1, 2, \dots, p\}$. For each

edge $\{x, y\}$, mark the label $f_e * (xy) = f_v(x) + f_v(y)$. f_v is named a parity sum cordial labeling (PSCLabeling), if f_v is a one-to-one function and $|e(0) - e(1)| \leq 1$, $e(1)$ and $e(0)$ represent the number of edges marked with odd numbers and even numbers respectively. A graph which admits a parity sum cordial labeling is called a Parity Sum Cordial graph (PSC graph).

Illustration 3.1

Consider the path graph P_5 . Let $a_1, a_2, \dots, a_5$ be the vertices of the path graph P_5 . Label the vertices of P_5 as shown in figure 1. Then the induced edge labels are 4, 5, 6, 9. Here the difference between number of edges marked with odd numbers and even numbers is less than or equal to one. Thus P_5 admits PSC labeling.

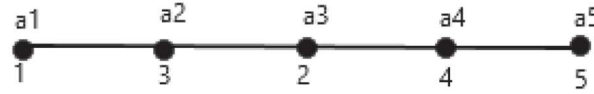


Figure 1

Theorem 3.1

Any path graph P_n is a Parity Sum Cordial Graph for all n belongs to $\mathbb{N}$.

Proof : Consider a path graph P_n with vertex set $V_1(P_n) = \{a_1, a_2, \dots, a_n\}$, and edge set $E_1(P_n) = \{e_1, e_2, \dots, e_{n-1}\}$ where $e_1 = \{a_1, a_2\}$, $e_2 = \{a_2, a_3\}$... $e_{n-1} = \{a_{n-1}, a_n\}$.

Here $p = n$ and $q = n - 1$

Define $f_v : V_1(P_n) \rightarrow \{1, 2, \dots, n\}$ as follows:

Case (i) : n is even and $n = 4i, i = 1, 2, 3, \dots$

$$f_v(a_1) = 1, f_v(a_2) = 3, f_v(a_3) = 2, f_v(a_4) = 4$$

.....

$$f_v(a_{n-1}) = n - 2, f_v(a_n) = n$$

Using this vertex labeling the resulting edge labels are

$$f_e * (e_1) = 4, f_e * (e_2) = 5, f_e * (e_3) = 6,$$

.....

$$f_e * (e_{n-1}) = n + n - 2 = 2n - 2.$$

Thus the resulting labels of edges are alternate between even and odd numbers.

Since P_n has $n - 1$ edges, $e(0) = \frac{n}{2}$ & $e(1) = \frac{n}{2}$.

Here $|e(0) - e(1)| \leq 1$.

Case (ii) : n is even and $n \neq 4i, i = 1, 2, 3, \dots$

$$f_v(a_1) = 1, f_v(a_2) = 3, f_v(a_3) = 2, f_v(a_4) = 4$$

.....

$$f_v(a_{n-1}) = n - 1, f_v(a_n) = n$$

Using this vertex labeling the resulting labels of edges are

$$f_e * (e_1) = 4, f_e * (e_2) = 5, f_e * (e_3) = 6,$$

... ..

$$f_e * (e_{n-2}) = n - 1 + n - 2 = 2n - 3,$$

$$f_e * (e_{n-1}) = n + n - 1 = 2n - 1.$$

Thus the resulting labels of edges are alternate between even and odd numbers from the beginning and last two are odd numbers.

Since P_n has $n-1$ edges, $e(1) = \frac{n}{2}$ & $e(0) = \frac{n}{2} - 1$.

Here $|e(0) - e(1)| \leq 1$.

Case (iii) : n is odd and $n = 4i - 1, i = 1, 2, 3, \dots$

$$f_v(a_1) = 1, f_v(a_2) = 3, f_v(a_3) = 2, f_v(a_4) = 4$$

.....

$$f_v(a_{n-1}) = n, f_v(a_n) = n - 1$$

Using this vertex labeling the resulting edge labels are

$$f_e * (e_1) = 4, f_e * (e_2) = 5, f_e * (e_3) = 6,$$

... ..

$$f_e * (e_{n-1}) = n + n - 1 = 2n - 1.$$

Thus the resulting labels of edges are alternate between even and odd numbers from the beginning and last two are odd numbers.

Since P_n has $n-1$ edges, $e(1) = \frac{n-1}{2}$ & $e(0) = \frac{n-1}{2}$.

Case (iv) : n is odd and $n \neq 4i - 1, i = 1, 2, 3, \dots$

$$f_v(a_1) = 1, f_v(a_2) = 3, f_v(a_3) = 2, f_v(a_4) = 4$$

.....

$$f_v(a_{n-1}) = n - 1, f_v(a_n) = n$$

Using this vertex labeling the induced edge labels are

$$f_e^*(e_1) = 4, f_e^*(e_2) = 5, f_e^*(e_3) = 6,$$

... ..

$$f_e^*(e_{n-1}) = n + n - 1 = 2n - 1.$$

Thus the resulting labels of edges are alternate between even and odd numbers from the beginning and last two are odd numbers.

Since P_n has $n-1$ edges, $e(1) = \frac{n-1}{2}$ & $e(0) = \frac{n-1}{2}$.

Here $|e(0) - e(1)| \leq 1$.

Hence in all the four cases the path graph P_n satisfies the condition of a PSC labeling.

Thus any path P_n is a PSC Graph.

Theorem 3.2

Cycle graphs C_n are PSC graph only when $n \neq 4i + 2, i = 1, 2, 3, \dots$ and $n \in N$.

Proof: Consider a cycle graph C_n with vertex set $V_1(C_n) = \{a_1, a_2, \dots, a_n\}$, and edge set $E_1(C_n) = \{e_1, e_2, \dots, e_{n-1}, e_n\}$ where $e_1 = \{a_1, a_2\}, e_2 = \{a_2, a_3\}, \dots, e_{n-1} =$

$$\{a_{n-1}, a_n\}, e_n = \{a_n, a_1\}.$$

Here $p = n$ and $q = n$

Case (i) : n is even and $n = 4i, i = 1, 2, 3, \dots$

Define $f_v: V_1(C_n) \rightarrow \{1, 2, \dots, n\}$ as follows:

$$f_v(a_1) = 1, f_v(a_2) = 3, f_v(a_3) = 2, f_v(a_4) = 4$$

.....

$$f_v(a_{n-1}) = n - 2, f_v(a_n) = n$$

Using this vertex labeling the resulting edge labels are

$$f_e * (e_1) = 4, f_e * (e_2) = 5, f_e * (e_3) = 6,$$

.....

$$f_e * (e_{n-1}) = n + n - 2 = 2n - 2.$$

Thus the resulting labels of edges are alternate between even and odd numbers.

Since C_n has n edges, $e(0) = \frac{n}{2}$ & $e(1) = \frac{n}{2}$

Here $|e(0) - e(1)| \leq 1$.

Case (ii) : n is odd, $n = 4i - 1, i = 1, 2, 3, \dots$

Define $f_v: V_1(C_n) \rightarrow \{1, 2, \dots, n\}$ as follows:

$$f_v(a_1) = 1, f_v(a_2) = 3, f_v(a_3) = 2, f_v(a_4) = 4$$

.....

$$f_v(a_{n-1}) = n - 2, f_v(a_n) = n$$

Using this vertex labeling the resulting labels of edges are alternate between even and odd numbers. Since C_n has n edges, $e(0) = \frac{n+1}{2}$ & $e(1) = \frac{n-1}{2}$.

Here $|e(0) - e(1)| \leq 1$.

Case (iii) : n is odd, $n = 4i - 1, i = 1, 2, 3, \dots$

Define $f_v: V_1(C_n) \rightarrow \{1, 2, \dots, n\}$ as follows:

$$f_v(a_1) = 1, f_v(a_2) = 3, f_v(a_3) = 2, f_v(a_4) = 4$$

.....

$$f_v(a_{n-1}) = n \text{ \& } f_v(a_n) = n - 1.$$

Using this vertex labeling the resulting labels of edges are alternate between even and odd numbers. Since C_n has n edges, $e(0) = \frac{n-1}{2}$ & $e(1) = \frac{n+1}{2}$

Here $|e(0) - e(1)| \leq 1$.

Case (iv) : n is even and $n = 4i + 2, i = 1, 2, 3, \dots$

For any two positive integers x and y , the sum $x + y$ is even if and only if x and y have the same parity; otherwise, $x + y$ is odd.

Let the numbers be $1, 2, \dots, 4i + 2$. Since n is even, the inequality $|e(0) - e(1)| \leq 1$ holds if and only if $e(0) = \frac{n}{2} = e(1)$. Hence, the numbers of even-labeled edges and odd-labeled edges must be equal. This is possible only if the $(4i+2)$ vertices are labelled in the alternating block pattern:

odd, odd, even, even, odd, odd, ... even, even

or even, even, odd, odd, even, even, ... odd, odd.

That is, the labeling consists of consecutive pairs of integers of the same parity, beginning with two odd integers and ending with two even integers, or vice versa. Such a circular pattern exists only when both the number of odd integers and the number of even integers are even. Since the set $\{1, 2, \dots, 4i+2\}$ contains $(2i+1)$ odd integers and $(2i+1)$ even integers, both of which are odd, no such arrangement is possible. Here $|e(0) - e(1)| > 1$.

Thus Cycles C_n are PSC graph only when $n \neq 4i+2, i = 1, 2, 3, \dots$

Theorem 3.3

The Comb $P_n \odot K_1$ is a Parity Sum Cordial Graph for all $n \in \mathbb{N}$.

Proof:

A comb graph $P_n \odot K_1$ is a special type of graph made by taking a path P_n and attaching a single extra vertex by an edge to every single vertex along the path P_n .

Here $p=2n$ and $q=2n-1$.

Define $f_v: V_1(P_n \odot K_1) \rightarrow \{1, 2, \dots, 2n\}$ as follows:

Label all the vertices of P_n with odd numbers and the pendent vertices with even numbers. Using this vertex labeling the resulting edge labels are odd for n edges and even for $n-1$ edges. Here $|e(0) - e(1)| \leq 1$.

Thus the Comb $P_n \odot K_1$ is a Parity Sum Cordial Graph.

Theorem 3.4

Complete graphs K_n are Parity Sum Cordial Graphs for all $n \in \mathbb{N}$ with $n \leq 3$.

Proof:

Consider a complete graph K_n with vertex set $V_1(K_n) = \{a_1, a_2, \dots, a_n\}$, and edge set $E_1(K_n) = \{e_1, e_2, \dots, e_{n(n-1)/2}\}$

Here $p = n$ and $q = \frac{n(n-1)}{2}$.

Define a bijective mapping $f_v: V_1(K_n) \rightarrow \{1, 2, \dots, n\}$ such that

$$f_v(a_1) = 1, f_v(a_2) = 2, f_v(a_3) = 3, f_v(a_4) = 4$$

.....

$$f_v(a_{n-1}) = n - 1 \text{ \& } f_v(a_n) = n.$$

The $\frac{n(n-1)}{2}$ edges of K_n are as follows:

$$\{a_1, a_2\}, \{a_1, a_3\}, \dots, \{a_1, a_n\}$$

$$\{a_2, a_3\}, \{a_2, a_4\}, \dots, \{a_2, a_n\}$$

$$\{a_3, a_4\}, \{a_3, a_5\}, \dots, \{a_3, a_n\}$$

... ..

$$\{a_{n-3}, a_{n-2}\}, \{a_{n-3}, a_{n-1}\}, \{a_{n-3}, a_n\}$$

$$\{a_{n-2}, a_{n-1}\}, \{a_{n-2}, a_n\}$$

$$\{a_{n-1}, a_n\}.$$

The labels of the corresponding end vertices of each edge are :

$$\{1, 2\}, \{1, 3\}, \dots, \{1, n\}$$

$$\{2, 3\}, \{2, 4\}, \dots, \{2, n\}$$

$$\{3, 4\}, \{3, 5\}, \dots, \{3, n\}$$

... ..

$$\{n - 3, n - 2\}, \{n - 3, n - 1\}, \{n - 3, n\}$$

$$\{n - 2, n - 1\}, \{n - 2, n\}$$

$$\{n - 1, n\}.$$

The resulting edge labels and the numbers of odd and even edge labels are as follows:

Table 1

Edge Labels	If n is even		If n is odd	
	Number of odd edge labels	Number of even edge labels	Number of odd edge labels	Number of even edge labels
3, 4, 5, ..., n + 1	$\frac{n}{2}$	$\frac{n}{2} - 1$	$\frac{n-1}{2}$	$\frac{n-1}{2}$
5, 6, 7, ..., n + 2	$\frac{n}{2} - 1$	$\frac{n}{2} - 1$	$\frac{n-1}{2}$	$\frac{n-1}{2} - 1$
7, 8, 9, ..., n + 3	$\frac{n}{2} - 1$	$\frac{n}{2} - 2$	$\frac{n-1}{2} - 1$	$\frac{n-1}{2} - 1$
.....				
2n - 5, 2n - 4, 2n - 3	2	1	2	1
2n - 3, 2n - 2	1	1	1	1
2n - 1	1	0	1	0
Total	$\frac{n^2}{4}$	$\frac{n^2}{4} - \frac{n}{2}$	$\frac{n^2-1}{4}$	$\frac{(n-1)^2}{4}$

Case (i): n is even

Number of odd edge labels, $e(1) = \frac{n^2}{4}$

Number of even edge labels, $e(0) = \frac{n^2}{4} - \frac{n}{2}$

Therefore $|e(0) - e(1)| \leq 1$ only when $n = 2$.

Case (ii): n is odd

Number of odd edge labels, $e(1) = \frac{n^2-1}{4}$

Number of even edge labels, $e(0) = \frac{(n-1)^2}{4}$

Therefore $|e(0) - e(1)| \leq 1$ only when $n = 3$ and $n = 1$.

Thus the complete graphs K_n are Parity Sum Cordial Graphs if and only if $n \leq 3$.

4. Application: Balanced Traffic Splitting Using PSC Labeling

4.1 Problem Setting

Consider a communication network with five routers connected in a linear topology. In order to prevent congestion, the network operator wants to divide the traffic equally between two communication channels, Channel A and Channel B, while data packets move between neighbouring routers.

- Vertices $V_1(G)$: devices (routers)
- Edges $E_1(G)$: communication links between devices

The network controller follows a simple routing policy that traffic is routed through Channel A if the edge label is even and traffic is routed through Channel B if the edge label is odd. The requirement is that both channels should carry almost equal load and avoid congestion or overuse of one channel. The communication network is represented by the path graph P_5 .

Step 1: Model the Network

Let the $V_1(P_5) = \{R_1, R_2, R_3, R_4, R_5\}$ where R_i represents the i^{th} router. The communication links are $E_1(P_5) = \{R_1R_2, R_2R_3, R_3R_4, R_4R_5\}$. Assign the routers the unique labels $f_v(R_1) = 1, f_v(R_2) = 3, f_v(R_3) = 2, f_v(R_4) = 4, f_v(R_5) = 5$.

Step 2: Classify each communication link

For each communication link $\{x,y\}$, $f_e^*(xy) = f_v(x) + f_v(y)$.

Table 2

Communication Link	End-vertex labels	Sum	Parity	Assigned Channel
R_1R_2	(1,3)	4	Even	Channel A
R_2R_3	(3,2)	5	Odd	Channel B
R_3R_4	(2,4)	6	Even	Channel A
R_4R_5	(4,5)	9	odd	Channel B

4.2 Practical Interpretations

Links in communication networks frequently transport comparable volumes of data. Congestion and delays may arise if a single channel is used for all traffic. Based on the parity of the sum of the router IDs, PSC-labeling offers a straightforward mathematical method for allocating each communication link to one of two channels. The communication load is automatically distributed between the two channels since the number of odd- and even-sum edges differs by no more than one. This establishes how PSC-labeling can be applied in communication networks as a lightweight trafficsplitting

technique that guarantees almost equitable usage of available transmission resources.

5. Conclusion

In this work we presented a new labeling technique called PSC labeling and its applications in communication networks. It converts a graph problem into a deterministic load-balancing mechanism. With this labeling balanced edge distribution can be maintained.

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