Mapana – Journal of Sciences
2026, Vol. 25, No. 2, *125 - 142*
ISSN 0975-3303 | https://doi.org/10.12723/mjs.77.7

A New Type of Nano Generalized Closed Sets in Nano Topological Spaces with Applications to Attribute Reduction in Student Dropout Risk

K. Muthu Maheswari[*] and D. Christia Jebakumari[*]

Abstract

This research study focuses on the notion of $\mathcal{N}_{\alpha}\mathcal{MC}$-closed sets, which play a significant role in the structural analysis of nano topological spaces and the development of generalized topological concepts. The $\mathcal{N}_{\alpha}\mathcal{MC}$-closed sets is introduced as a new class of nano generalized closed sets formulated within the framework of nano topology. In this study, the concept of $\mathcal{N}_{\alpha}\mathcal{MC}$-closed sets is defined and investigated through its fundamental properties and characteristic features. The relationships between $\mathcal{N}_{\alpha}\mathcal{MC}$-closed sets and several existing classes of nano closed sets are also established and analyzed. Moreover, the notions of $\mathcal{N}_{\alpha}\mathcal{MC}$-interior and $\mathcal{N}_{\alpha}\mathcal{MC}$-closure are characterized using $\mathcal{N}_{\alpha}\mathcal{MC}$-closed sets, highlighting their structural behavior and topological significance within nano topological spaces. Further, the concept is applied to attribute reduction in student dropout risk analysis in higher education. By using nano topological bases, the key attributes affecting dropout risk are identified. The study shows that the proposed method can be effectively used for decision-making problems.

Keywords: $\mathcal{N}_{\alpha}\mathcal{MC}$-closed sets, $\mathcal{N}_{\alpha}\mathcal{MC}$-open sets, $\mathcal{N}_{\alpha}\mathcal{MC}$-interior, $\mathcal{N}_{\alpha}\mathcal{MC}$-closure, attributes, core.

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[*] Department of Mathematics and Research Centre, Sarah Tucker College (Autonomous), Tirunelveli, Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli, Tamil Nadu, India; kmmaheswari97 @gmail.com, christiajeba@gmail.com

1. Introduction

Nano topology, introduced by M. L. Thivagar and C. Richard [5], has emerged as an important area in topological studies. This concept was formulated using lower and upper approximations together with the boundary region of a subset of a universe under an equivalence relation. Following this development, several researchers contributed to the advancement of nano topological structures. In 2014, K. Bhuvaneswari and K. Mythili Gnanapriya [2] introduced nano generalized closed sets and examined their fundamental properties in nano topological spaces. Later, in 2015, V. Rajendran, P. Sathishmohan, and K. Indirani [10] proposed nano generalized star closed sets and investigated their significant characteristics. In 2016, M. Parimala, C. Indirani, and S. Jafari [9] introduced nano b-open sets. Subsequently, in 2022, P. Anbarasi Rodrigo and P. Subithra [1] developed the notion of nano generalized α^*-closed sets. R. Maheswari and P. Lavanya [7] applied nano topological concepts to attribute reduction problems in their study on the elimination of attributes in peptic ulcers in human beings. Inspired by recent developments in nano topology and the study of nano α-closed and nano b-closed sets, this paper introduces and studies $\mathcal{N}_\alpha\mathcal{MC}$-closed sets, explores their relationships with existing nano closed sets, and establishes related operators such as $\mathcal{N}_\alpha\mathcal{MC}$-interior and $\mathcal{N}_\alpha\mathcal{MC}$-closure using $\mathcal{N}_\alpha\mathcal{MC}$-closed and $\mathcal{N}_\alpha\mathcal{MC}$-open sets. Finally, the proposed concepts are applied to an attribute reduction problem in student dropout risk analysis.

2. Preliminaries

Definition 2.1 [4] Let $\mathcal{U}$ be a non-empty finite set of objects called the universe and $\mathcal{R}$ be an equivalence relation on $\mathcal{U}$ named as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair $(\mathcal{U}, \mathcal{R})$ is said to be the approximation space. Let $\mathcal{K} \subseteq \mathcal{U}$. Then

1. The lower approximation of $\mathcal{K}$ with respect to $\mathcal{R}$ is denoted by $L_{\mathcal{R}}(\mathcal{K})$. That is, $L_{\mathcal{R}}(\mathcal{K}) = \cup_{k\in\mathcal{U}}\{\mathcal{R}(k): \mathcal{R}(k) \subseteq \mathcal{K}\}$, where $\mathcal{R}(k)$ denotes the equivalence class determined by k.

2. The upper approximation of $\mathcal{K}$ with respect to $\mathcal{R}$ is denoted by $U_{\mathcal{R}}(\mathcal{K})$. That is, $U_{\mathcal{R}}(\mathcal{K}) = \cup_{k\in\mathcal{U}}\{\mathcal{R}(k): \mathcal{R}(k) \cap \mathcal{K} \neq \emptyset\}$.

3. The boundary region of $\mathcal{K}$ with respect to $\mathcal{R}$ is the set of all objects, which can be classified neither as $\mathcal{K}$ nor as not-$\mathcal{K}$ with respect to $\mathcal{R}$ and it is denoted by $B_{\mathcal{R}}(\mathcal{K})$. That is $B_{\mathcal{R}}(\mathcal{K}) = U_{\mathcal{R}}(\mathcal{K}) - L_{\mathcal{R}}(\mathcal{K})$.

Definition 2.2 [4] Let $\mathcal{U}$ be an universal set and $\mathcal{R}$ be an equivalence relation on $\mathcal{U}$ and $\tau_{\mathcal{R}}(\mathcal{K}) = \{\mathcal{U}, \phi, L_{\mathcal{R}}(\mathcal{K}), U_{\mathcal{R}}(\mathcal{K}), B_{\mathcal{R}}(\mathcal{K})\}, \mathcal{K} \subseteq \mathcal{U}$. Then $\tau_{\mathcal{R}}(\mathcal{K})$ satisfies the following axioms:

i. $\mathcal{U}, \emptyset \in \tau_{\mathcal{R}}(\mathcal{K})$.
ii. The union of the elements of any subcollection of $\tau_{\mathcal{R}}(\mathcal{K})$ is in $\tau_{\mathcal{R}}(\mathcal{K})$.
iii. The intersection of the elements of any finite subcollection of $\tau_{\mathcal{R}}(\mathcal{K})$ is in $\tau_{\mathcal{R}}(\mathcal{K})$.

Then $\tau_{\mathcal{R}}(\mathcal{K})$ is a topology on $\mathcal{U}$ is called the nano topology on $\mathcal{U}$ with respect to $\mathcal{K}$ and $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{K}))$ is called the nano topological space (briefly $\mathcal{N}_{\mathfrak{a}}\mathcal{T}$ space). The elements of $\tau_{\mathcal{R}}(\mathcal{K})$ are called nano open sets (simply, $\mathcal{N}_{\mathfrak{a}}\mathfrak{os}$) and the complement of nano open sets are called nano closed sets (simply, $\mathcal{N}_{\mathfrak{a}}\mathfrak{cs}$).

Definition 2.3 [4] If $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{K}))$ is the $\mathcal{N}_{\mathfrak{a}}\mathcal{T}$space with respect to $\mathcal{K}$ where $\mathcal{K} \subseteq \mathcal{U}$ and if $\mathcal{P} \subseteq \mathcal{U}$, then

1. The nano interior of a set $\mathcal{P}$ (simply, $\mathcal{N}_{\mathfrak{a}}int(\mathcal{P})$) is described as the union of all $\mathcal{N}_{\mathfrak{a}}\mathfrak{os}$ contained in $\mathcal{P}$. $\mathcal{N}_{\mathfrak{a}}int(\mathcal{P})$ is the largest $\mathcal{N}_{\mathfrak{a}}$open subset of $\mathcal{P}$.
2. The nano closure of a set $\mathcal{P}$ (simply, $\mathcal{N}_{\mathfrak{a}}cl(\mathcal{P})$) is described as the intersection of all $\mathcal{N}_{\mathfrak{a}}\mathfrak{cs}$ containing $\mathcal{P}$. $\mathcal{N}_{\mathfrak{a}}cl(\mathcal{P})$ is the smallest $\mathcal{N}_{\mathfrak{a}}$closed set containing $\mathcal{P}$.

Definition 2.4 [4] If $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{K}))$ is the $\mathcal{N}_{\mathfrak{a}}\mathcal{T}$space with respect to $\mathcal{K}$ where $\mathcal{K} \subseteq \mathcal{U}$ and if $\mathcal{P} \subseteq \mathcal{U}$, then

1. The nano interior* of a set $\mathcal{P}$ (simply, $\mathcal{N}_{\mathfrak{a}}int^{*}(\mathcal{P})$) is described as the union of all $\mathcal{N}_{\mathfrak{a}}g\mathfrak{os}$ contained in $\mathcal{P}$.
2. The nano closure* of a set $\mathcal{P}$ (simply, $\mathcal{N}_{\mathfrak{a}}cl^{*}(\mathcal{P})$) is described as the intersection of all $\mathcal{N}_{\mathfrak{a}}g\mathfrak{cs}$ containing $\mathcal{P}$.

Definition 2.5 [7] A subset $\mathcal{P}$ of a $\mathcal{N}_{\mathfrak{a}}\mathcal{T}$space $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{K}))$ is called

(i) Nano pre-open set (simply, $\mathcal{N}_{\mathfrak{a}}p\mathfrak{os}$) if $\mathcal{P} \subseteq \mathcal{N}_{\mathfrak{a}}int(\mathcal{N}_{\mathfrak{a}}cl(\mathcal{P}))$.
(ii) Nano semi-open set (simply, $\mathcal{N}_{\mathfrak{a}}s\mathfrak{os}$) if $\mathcal{P} \subseteq \mathcal{N}_{\mathfrak{a}}cl(\mathcal{N}_{\mathfrak{a}}int(\mathcal{P}))$.
(iii) Nano α-open set (simply, $\mathcal{N}_{\mathfrak{a}}\alpha\mathfrak{os}$) if $\mathcal{P} \subseteq \mathcal{N}_{\mathfrak{a}}int(\mathcal{N}_{\mathfrak{a}}cl(\mathcal{N}_{\mathfrak{a}}int(\mathcal{P})))$.
(iv) Nano regular-open set (simply, $\mathcal{N}_{\mathfrak{a}}r\mathfrak{os}$) if $\mathcal{P} = \mathcal{N}_{\mathfrak{a}}int(\mathcal{N}_{\mathfrak{a}}cl(\mathcal{P}))$.
(v) Nano b -open set (simply, $\mathcal{N}_{\mathfrak{a}}b\mathfrak{os}$) if $\mathcal{P} \subseteq \mathcal{N}_{\mathfrak{a}}int(\mathcal{N}_{\mathfrak{a}}cl(\mathcal{P})) \cup \mathcal{N}_{\mathfrak{a}}cl(\mathcal{N}_{\mathfrak{a}}int(\mathcal{P}))$.

The complements of the above-mentioned sets are called their respective $\mathcal{N}_{\mathfrak{a}}$closed sets.

Definition 2.6 [5,8,9,10] A subset $\mathcal{P}$ of a $\mathcal{N}_{\mathfrak{a}}\mathcal{T}$space $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{K}))$ is called

(i) $\mathcal{N}_{\mathfrak{a}}g$-closed set (simply, $\mathcal{N}_{\mathfrak{a}}g\mathfrak{cs}$) if $\mathcal{N}_{\mathfrak{a}}cl(\mathcal{P}) \subseteq \mathcal{S}$, whenever $\mathcal{P} \subseteq \mathcal{S}$ and $\mathcal{S}$ is $\mathcal{N}_{\mathfrak{a}}\mathfrak{os}$ in $\mathcal{U}$.

(ii) $\mathcal{N}_{\mathfrak{a}}g^*$-closed set (simply, $\mathcal{N}_{\mathfrak{a}}g^*\mathfrak{cs}$) if $\mathcal{N}_{\mathfrak{a}}cl(\mathcal{P}) \subseteq \mathcal{S}$, whenever $\mathcal{P} \subseteq \mathcal{S}$ and $\mathcal{S}$ is $\mathcal{N}_{\mathfrak{a}}g\mathfrak{os}$ in $\mathcal{U}$.

(iii) $\mathcal{N}_{\mathfrak{a}}g^{**}$-closed set (simply, $\mathcal{N}_{\mathfrak{a}}g^{**}\mathfrak{cs}$) if $\mathcal{N}_{\mathfrak{a}}cl(\mathcal{P}) \subseteq \mathcal{S}$, whenever $\mathcal{P} \subseteq \mathcal{S}$ and $\mathcal{S}$ is $\mathcal{N}_{\mathfrak{a}}g^*\mathfrak{os}$ in $\mathcal{U}$.

(iv) $\mathcal{N}_{\mathfrak{a}}gb$-closed set (simply, $\mathcal{N}_{\mathfrak{a}}gb\mathfrak{cs}$) if $\mathcal{N}_{\mathfrak{a}}bcl(\mathcal{P}) \subseteq \mathcal{S}$, whenever $\mathcal{P} \subseteq \mathcal{S}$ and $\mathcal{S}$ is $\mathcal{N}_{\mathfrak{a}}\mathfrak{os}$ in $\mathcal{U}$.

(v) $\mathcal{N}_{\mathfrak{a}}g^*b$-closed set (simply, $\mathcal{N}_{\mathfrak{a}}g^*b\mathfrak{cs}$) if $\mathcal{N}_{\mathfrak{a}}bcl(\mathcal{P}) \subseteq \mathcal{S}$, whenever $\mathcal{P} \subseteq \mathcal{S}$ and $\mathcal{S}$ is $\mathcal{N}_{\mathfrak{a}}g\mathfrak{os}$ in $\mathcal{U}$.

Remark 2.7 [8]

(i) Every $\mathcal{N}_{\mathfrak{a}}\mathfrak{os}$ is $\mathcal{N}_{\mathfrak{a}}g^*\mathfrak{os}$.

(ii) Every $\mathcal{N}_{\mathfrak{a}}g^*\mathfrak{os}$ is $\mathcal{N}_{\mathfrak{a}}g\mathfrak{os}$.

3. $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}$-Closed Sets

Definition 3.1 Let $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{K}))$ be a $\mathcal{N}_{\mathfrak{a}}\mathcal{T}$ space. A subset $\mathcal{P}$ of a $\mathcal{N}_{\mathfrak{a}}\mathcal{T}$ space $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{K}))$ is called $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}$-closed set (simply, $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}\mathfrak{cs}$) if $\mathcal{N}_{\mathfrak{a}}bcl(\mathcal{P}) \subset \mathcal{N}_{\mathfrak{a}}int^*(\mathcal{S})$ whenever $\mathcal{P} \subset \mathcal{S}$ and $\mathcal{S}$ is $\mathcal{N}_{\mathfrak{a}}g^*$-open in $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{K}))$.

Example 3.2 Let $\mathcal{U} = \{e, f, g, h\}$ with $\mathcal{U}/\mathcal{R} = \big\{\{e\}, \{g\}, \{f, h\}\big\}$ and $\mathcal{K} = \{e, h\}$. Then $\tau_{\mathcal{R}}(\mathcal{K}) = \{\emptyset, \{e\}, \{f, h\}, \{e, f, h\}, \mathcal{U}\}$. Here
$\emptyset, \{e\}, \{f\}, \{g\}, \{h\}, \{e, g\}, \{f, g\}, \{f, h\}, \{g, h\}$,

$\{e, f, g\}, \{e, g, h\}, \{f, g, h\}, \mathcal{U}$ are $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}$-closed sets.

Theorem 3.3 Every $\mathcal{N}_{\mathfrak{a}}\mathfrak{cs}$ is $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}\mathfrak{cs}$.

Proof: Consider an $\mathcal{N}_{\mathfrak{a}}\mathfrak{cs}$ $\mathcal{P}$ in $\mathcal{N}_{\mathfrak{a}}\mathcal{T}$ space $\big(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{K})\big)$. For a $\mathcal{N}_{\mathfrak{a}}\mathfrak{os}$ $\mathcal{S}$ in $\big(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{K})\big)$ with $\mathcal{P} \subset \mathcal{S}$, $\mathcal{N}_{\mathfrak{a}}int(\mathcal{S}) = \mathcal{S}$. By Remark 2.7(i), $\mathcal{S}$ is $\mathcal{N}_{\mathfrak{a}}g^*\mathfrak{os}$. Since $\mathcal{P}$ is $\mathcal{N}_{\mathfrak{a}}\mathfrak{cs}$, $\mathcal{N}_{\mathfrak{a}}bcl(\mathcal{P}) \subset \mathcal{N}_{\mathfrak{a}}cl(\mathcal{P}) = \mathcal{P} \subset \mathcal{S} = \mathcal{N}_{\mathfrak{a}}int(\mathcal{S}) \subset \mathcal{N}_{\mathfrak{a}}int^*(\mathcal{S})$. Therefore $\mathcal{N}_{\mathfrak{a}}bcl(\mathcal{P}) \subset \mathcal{N}_{\mathfrak{a}}int^*(\mathcal{S})$ whenever $\mathcal{P} \subset \mathcal{S}$ and $\mathcal{S}$ is $\mathcal{N}_{\mathfrak{a}}g^*\mathfrak{os}$ in $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{K}))$. Thus $\mathcal{P}$ is $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}\mathfrak{cs}$.

Remark 3.4 Although the Theorem 3.3 holds, its converse fails in general, as shown by counterexample 3.5.

Example 3.5 Let $\mathcal{U} = \{e, f, g\}$ with $\mathcal{U}/\mathcal{R} = \{\{e\}, \{f, g\}\}$ and $\mathcal{K} = \{e\}$.

Then $\tau_{\mathcal{R}}(\mathcal{K}) = \{\emptyset, \{e\}, \mathcal{U}\}$. Here $\{f\}$ is $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}\mathfrak{cs}$ but not $\mathcal{N}_{\mathfrak{a}}\mathfrak{cs}$.

Theorem 3.6 Every $\mathcal{N}_{\mathfrak{a}}p\mathfrak{cs}$ is $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}\mathfrak{cs}$.

Proof: Consider an $\mathcal{N}_a pcs$ $\mathcal{P}$ in $\mathcal{N}_a\mathcal{T}$ space $\left(\mathcal{U},\tau_{\mathcal{R}}(\mathcal{K})\right)$. For a $\mathcal{N}_a os$ $\mathcal{S}$ in $\left(\mathcal{U},\tau_{\mathcal{R}}(\mathcal{K})\right)$ with $\mathcal{P}\subset\mathcal{S}$, $\mathcal{N}_a int(\mathcal{S})=\mathcal{S}$. By Remark 2.7(i), $\mathcal{S}$ is $\mathcal{N}_a g^* os$. Since $\mathcal{P}$ is $\mathcal{N}_a pcs$, $\mathcal{N}_a bcl(\mathcal{P})\subset\mathcal{N}_a pcl(\mathcal{P})=\mathcal{P}\subset\mathcal{S}=\mathcal{N}_a int(\mathcal{S})\subset\mathcal{N}_a int^*(\mathcal{S})$. Therefore $\mathcal{N}_a bcl(\mathcal{P})\subset\mathcal{N}_a int^*(\mathcal{S})$ whenever $\mathcal{P}\subset\mathcal{S}$ and $\mathcal{S}$ is $\mathcal{N}_a g^* os$ in $(\mathcal{U},\tau_{\mathcal{R}}(\mathcal{K}))$. Thus $\mathcal{P}$ is $\mathcal{N}_a\mathcal{MC}cs$.

Remark 3.7 Although the Theorem 3.6 holds, its converse fails in general, as shown by counterexample 3.8.

Example 3.8 Let $\mathcal{U}=\{e,f,g,h\}$ with $\mathcal{U}/\mathcal{R}=\{\{e,f\},\{g\},\{h\}\}$ and $\mathcal{K}=\{f,g\}$. Then $\tau_{\mathcal{R}}(\mathcal{K})=\{\emptyset,\{g\},\{e,f\},\{e,f,g\},\mathcal{U}\}$. Here $\{e,f\}$ is $\mathcal{N}_a\mathcal{MC}cs$ but not $\mathcal{N}_a pcs$.

Theorem 3.9 Every $\mathcal{N}_a scs$ is $\mathcal{N}_a\mathcal{MC}cs$.

Proof: Consider an $\mathcal{N}_a scs$ $\mathcal{P}$ in $\mathcal{N}_a\mathcal{T}$ space $\left(\mathcal{U},\tau_{\mathcal{R}}(\mathcal{K})\right)$. For a $\mathcal{N}_a os$ $\mathcal{S}$ in $\left(\mathcal{U},\tau_{\mathcal{R}}(\mathcal{K})\right)$ with $\mathcal{P}\subset\mathcal{S}$, $\mathcal{N}_a int(\mathcal{S})=\mathcal{S}$. By Remark 2.7(i), $\mathcal{S}$ is $\mathcal{N}_a g^* os$. Since $\mathcal{P}$ is $\mathcal{N}_a scs$, $\mathcal{N}_a bcl(\mathcal{P})\subset\mathcal{N}_a scl(\mathcal{P})=\mathcal{P}\subset\mathcal{S}=\mathcal{N}_a int(\mathcal{S})\subset\mathcal{N}_a int^*(\mathcal{S})$. Therefore $\mathcal{N}_a bcl(\mathcal{P})\subset\mathcal{N}_a int^*(\mathcal{S})$ whenever $\mathcal{P}\subset\mathcal{S}$ and $\mathcal{S}$ is $\mathcal{N}_a g^* os$ in $(\mathcal{U},\tau_{\mathcal{R}}(\mathcal{K}))$. Thus $\mathcal{P}$ is $\mathcal{N}_a\mathcal{MC}cs$.

Remark 3.10 Although the Theorem 3.9 holds, its converse fails in general, as shown by counterexample 3.11.

Example 3.11 Let $\mathcal{U}=\{e,f,g,h\}$ with $\mathcal{U}/\mathcal{R}=\{\{e,f\},\{g\},\{h\}\}$ and $\mathcal{K}=\{f,g\}$. Then $\tau_{\mathcal{R}}(\mathcal{K})=\{\emptyset,\{g\},\{e,f\},\{e,f,g\},\mathcal{U}\}$. Here $\{f,h\}$ is $\mathcal{N}_a\mathcal{MC}cs$ but not $\mathcal{N}_a scs$.

Theorem 3.12 Every $\mathcal{N}_a\alpha cs$ is $\mathcal{N}_a\mathcal{MC}cs$.

Proof: Consider an $\mathcal{N}_a\alpha cs$ $\mathcal{P}$ in $\mathcal{N}_a\mathcal{T}$ space $\left(\mathcal{U},\tau_{\mathcal{R}}(\mathcal{K})\right)$. For a $\mathcal{N}_a os$ $\mathcal{S}$ in $\left(\mathcal{U},\tau_{\mathcal{R}}(\mathcal{K})\right)$ with $\mathcal{P}\subset\mathcal{S}$, $\mathcal{N}_a int(\mathcal{S})=\mathcal{S}$. By Remark 2.7(i), $\mathcal{S}$ is $\mathcal{N}_a g^* os$. Since $\mathcal{P}$ is $\mathcal{N}_a\alpha cs$, $\mathcal{N}_a bcl(\mathcal{P})\subset\mathcal{N}_a\alpha cl(\mathcal{P})=\mathcal{P}\subset\mathcal{S}=\mathcal{N}_a int(\mathcal{S})\subset\mathcal{N}_a int^*(\mathcal{S})$. Therefore $\mathcal{N}_a bcl(\mathcal{P})\subset\mathcal{N}_a int^*(\mathcal{S})$ whenever $\mathcal{P}\subset\mathcal{S}$ and $\mathcal{S}$ is $\mathcal{N}_a g^* os$ in $(\mathcal{U},\tau_{\mathcal{R}}(\mathcal{K}))$. Thus $\mathcal{P}$ is $\mathcal{N}_a\mathcal{MC}cs$.

Remark 3.13 Although the Theorem 3.12 holds, its converse fails in general, as shown by counterexample 3.14.

Example 3.14 Let $\mathcal{U}=\{e,f,g,h\}$ with $\mathcal{U}/\mathcal{R}=\{\{e\},\{f\},\{g,h\}\}$ and $\mathcal{K}=\{e,h\}$. Then $\tau_{\mathcal{R}}(\mathcal{K})=\{\emptyset,\{e\},\{g,h\},\{e,g,h\},\mathcal{U}\}$. Here $\{e,g\}$ is $\mathcal{N}_a\mathcal{MC}cs$ but not $\mathcal{N}_a\alpha cs$.

Theorem 3.15 Every $\mathcal{N}_a rcs$ is $\mathcal{N}_a\mathcal{MC}cs$.

Proof: Consider an $\mathcal{N}_a rcs$ $\mathcal{P}$ in $\mathcal{N}_a\mathcal{T}$ space $\left(\mathcal{U},\tau_{\mathcal{R}}(\mathcal{K})\right)$. For a $\mathcal{N}_a os$ $\mathcal{S}$ in $\left(\mathcal{U},\tau_{\mathcal{R}}(\mathcal{K})\right)$ with $\mathcal{P}\subset\mathcal{S}$, $\mathcal{N}_a int(\mathcal{S})=\mathcal{S}$. By Remark 2.7(i), $\mathcal{S}$ is $\mathcal{N}_a g^* os$. Since every $\mathcal{N}_{\mathfrak{a}}r\mathfrak{cs}$ is $\mathcal{N}_{\mathfrak{a}}\mathfrak{cs}$. Therefore $\mathcal{P}$ is $\mathcal{N}_{\mathfrak{a}}\mathfrak{cs}$. By theorem 3.3, $\mathcal{P}$ is $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}\mathfrak{cs}$.

Remark 3.16 Although the Theorem 3.15 holds, its converse fails in general, as shown by counterexample 3.17.

Example 3.17 Let $\mathcal{U} = \{e, f, g, h\}$ with $\mathcal{U}/\mathcal{R} = \{\{e\}, \{g\}, \{f, h\}\}$ and $\mathcal{K} = \{e, h\}$. Then $\tau_{\mathcal{R}}(\mathcal{K}) = \{\emptyset, \{e\}, \{f, h\}, \{e, f, h\}, \mathcal{U}\}$. Here $\{f, g\}$ is $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}\mathfrak{cs}$ but not $\mathcal{N}_{\mathfrak{a}}r\mathfrak{cs}$.

Theorem 3.18 Every $\mathcal{N}_{\mathfrak{a}}g\mathfrak{cs}$ is $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}\mathfrak{cs}$.

Proof: Consider an $\mathcal{N}_{\mathfrak{a}}g\mathfrak{cs}$ $\mathcal{P}$ in $\mathcal{N}_{\mathfrak{a}}\mathcal{T}$ space $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{K}))$. For a $\mathcal{N}_{\mathfrak{a}}\mathfrak{os}$ $\mathcal{S}$ in $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{K}))$ with $\mathcal{P} \subset \mathcal{S}$, $\mathcal{N}_{\mathfrak{a}}int(\mathcal{S}) = \mathcal{S}$. Since $\mathcal{P}$ is $g\mathcal{N}_{\mathfrak{a}}g\mathfrak{cs}$, $\mathcal{N}_{\mathfrak{a}}cl(\mathcal{P}) \subset \mathcal{S}$ whenever $\mathcal{PP} \subset \mathcal{S}$ and $\mathcal{S}$ is $\mathcal{N}_{\mathfrak{a}}\mathfrak{os}$. By Remark 2.7(i), $\mathcal{S}$ is $\mathcal{N}_{\mathfrak{a}}g^*\mathfrak{os}$. Since every $\mathcal{N}_{\mathfrak{a}}\mathfrak{cs}$ is $\mathcal{N}_{\mathfrak{a}}b\mathfrak{cs}$, $\mathcal{N}_{\mathfrak{a}}bcl(\mathcal{P}) \subset \mathcal{N}_{\mathfrak{a}}cl(\mathcal{P}) \subset \mathcal{S} = \mathcal{N}_{\mathfrak{a}}int(\mathcal{S}) \subset \mathcal{N}_{\mathfrak{a}}int^*(\mathcal{S})$. Therefore $\mathcal{N}_{\mathfrak{a}}bcl(\mathcal{P}) \subset \mathcal{N}_{\mathfrak{a}}int^*(\mathcal{S})$ whenever $\mathcal{P} \subset \mathcal{S}$ and $\mathcal{S}$ is $\mathcal{N}_{\mathfrak{a}}g^*\mathfrak{os}$ in $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{K}))$. Thus $\mathcal{P}$ is $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}\mathfrak{cs}$.

Remark 3.19 Although the Theorem 3.18 holds, its converse fails in general, as shown by counterexample 3.20.

Example 3.20 Let $\mathcal{U} = \{e, f, g, h\}$ with $\mathcal{U}/\mathcal{R} = \{\{e\}, \{h\}, \{f, g\}\}$ and $\mathcal{K} = \{g, h\}$. Then $\tau_{\mathcal{R}}(\mathcal{K}) = \{\emptyset, \{h\}, \{f, g\}, \{f, g, h\}, \mathcal{U}\}$. Here $\{f\}$ is $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}\mathfrak{cs}$ but not $\mathcal{N}_{\mathfrak{a}}g\mathfrak{cs}$.

Theorem 3.21 Every $\mathcal{N}_{\mathfrak{a}}g^{**}\mathfrak{cs}$ is $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}\mathfrak{cs}$.

Proof: Consider an $\mathcal{N}_{\mathfrak{a}}g^{**}\mathfrak{cs}$ $\mathcal{P}$ in $\mathcal{N}_{\mathfrak{a}}\mathcal{T}$ space $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{K}))$. For a $\mathcal{N}_{\mathfrak{a}}\mathfrak{os}$ $\mathcal{S}$ in $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{K}))$ with $\mathcal{P} \subset \mathcal{S}$, $\mathcal{N}_{\mathfrak{a}}int(\mathcal{S}) = \mathcal{S}$. By Remark 2.7(i), $\mathcal{S}$ is $\mathcal{N}_{\mathfrak{a}}g^*\mathfrak{os}$. Since $\mathcal{P}$ is $\mathcal{N}_{\mathfrak{a}}g^{**}\mathfrak{cs}$, $\mathcal{N}_{\mathfrak{a}}cl(\mathcal{P}) \subset \mathcal{S}$ whenever $\mathcal{P} \subset \mathcal{S}$ and $\mathcal{S}$ is $\mathcal{N}_{\mathfrak{a}}g^*\mathfrak{os}$. Since every $\mathcal{N}_{\mathfrak{a}}\mathfrak{cs}$ is $\mathcal{N}_{\mathfrak{a}}b\mathfrak{cs}$, $\mathcal{N}_{\mathfrak{a}}bcl(\mathcal{P}) \subset \mathcal{N}_{\mathfrak{a}}cl(\mathcal{P}) \subset \mathcal{S} = \mathcal{N}_{\mathfrak{a}}int(\mathcal{S}) \subset \mathcal{N}_{\mathfrak{a}}int^*(\mathcal{S})$. Therefore $\mathcal{N}_{\mathfrak{a}}bcl(\mathcal{P}) \subset \mathcal{N}_{\mathfrak{a}}int^*(\mathcal{S})$ whenever $\mathcal{P} \subset \mathcal{S}$ and $\mathcal{S}$ is $\mathcal{N}_{\mathfrak{a}}g^*\mathfrak{os}$ in $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{K}))$. Thus $\mathcal{P}$ is $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}\mathfrak{cs}$.

Remark 3.22 Although the Theorem 3.21 holds, its converse fails in general, as shown by counterexample 3.23.

Example 3.23 Let $\mathcal{U} = \{e, f, g, h\}$ with $\mathcal{U}/\mathcal{R} = \{\{e\}, \{g\}, \{f, h\}\}$ and $\mathcal{K} = \{e, h\}$. Then $\tau_{\mathcal{R}}(\mathcal{K}) = \{\emptyset, \{e\}, \{f, h\}, \{e, f, h\}, \mathcal{U}\}$. Here $\{f, h\}$ is $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}\mathfrak{cs}$ but not $\mathcal{N}_{\mathfrak{a}}g^{**}\mathfrak{cs}$.

Theorem 3.24 Every $\mathcal{N}_{\mathfrak{a}}g^*\mathfrak{cs}$ is $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}\mathfrak{cs}$.

Proof: Consider an $\mathcal{N}_{\mathfrak{a}}g^*\mathfrak{cs}$ $\mathcal{P}$ in $\mathcal{N}_{\mathfrak{a}}\mathcal{T}$ space $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{K}))$. For a $\mathcal{N}_{\mathfrak{a}}\mathfrak{os}$ $\mathcal{S}$ in $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{K}))$ with $\mathcal{P} \subset \mathcal{S}$, $\mathcal{N}_{\mathfrak{a}}int(\mathcal{S}) = \mathcal{S}$. By Remark 2.7(i), $\mathcal{S}$ is $\mathcal{N}_{\mathfrak{a}}g^*\mathfrak{os}$. Since every $\mathcal{N}_{\mathfrak{a}}g^*\mathfrak{cs}$ is a $\mathcal{N}_{\mathfrak{a}}g^{**}\mathfrak{cs}$. Therefore $\mathcal{P}$ is $\mathcal{N}_{\mathfrak{a}}g^{**}\mathfrak{cs}$. By theorem 3.21, $\mathcal{P}$ is $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}\mathfrak{cs}$.

Remark 3.25 Although the Theorem 3.24 holds, its converse fails in general, as shown by counterexample 3.26.

Example 3.26 Let $\mathcal{U} = \{e, f, g\}$ with $\mathcal{U}/\mathcal{R} = \{\{e, f\}, \{g\}\}$ and $\mathcal{K} = \{f\}$. Then $\tau_{\mathcal{R}}(\mathcal{K}) = \{\emptyset, \{e, f\}, \mathcal{U}\}$. Here $\{e\}$ is $\mathcal{N}_{\alpha}\mathcal{MC}\mathfrak{cs}$ but not $\mathcal{N}_{\alpha}g^{*}\mathfrak{cs}$.

Theorem 3.27 Every $\mathcal{N}_{\alpha}gb\mathfrak{cs}$ is $\mathcal{N}_{\alpha}\mathcal{MC}\mathfrak{cs}$.

Proof: Consider an $\mathcal{N}_{\alpha}gb\mathfrak{cs}$ $\mathcal{P}$ in $\mathcal{N}_{\alpha}\mathcal{T}$ space $\big(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{K})\big)$. For a $\mathcal{N}_{\alpha}\mathfrak{os}$ $\mathcal{S}$ in $\big(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{K})\big)$ with $\mathcal{P} \subset \mathcal{S}$, $\mathcal{N}_{\alpha}int(\mathcal{S}) = \mathcal{S}$. Since $\mathcal{P}$ is $\mathcal{N}_{\alpha}gb\mathfrak{cs}$, $\mathcal{N}_{\alpha}bcl(\mathcal{P}) \subset \mathcal{S}$ whenever $\mathcal{P} \subset \mathcal{S}$ and $\mathcal{S}$ is $\mathcal{N}_{\alpha}\mathfrak{os}$. By Remark 2.7(i), $\mathcal{S}$ is $\mathcal{N}_{\alpha}g^{*}\mathfrak{os}$. Then $\mathcal{N}_{\alpha}bcl(\mathcal{P}) \subset \mathcal{S} = \mathcal{N}_{\alpha}int(\mathcal{S}) \subset \mathcal{N}_{\alpha}int^{*}(\mathcal{S})$. Therefore $\mathcal{N}_{\alpha}bcl(\mathcal{P}) \subset \mathcal{N}_{\alpha}int^{*}(\mathcal{S})$ whenever $\mathcal{P} \subset \mathcal{S}$ and $\mathcal{S}$ is $\mathcal{N}_{\alpha}g^{*}\mathfrak{os}$ in $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{K}))$. Thus $\mathcal{P}$ is $\mathcal{N}_{\alpha}\mathcal{MC}\mathfrak{cs}$.

Remark 3.28 Although the Theorem 3.27 holds, its converse fails in general, as shown by counterexample 3.29.

Example 3.29 Let $\mathcal{U} = \{e, f, g, h\}$ with $\mathcal{U}/\mathcal{R} = \big\{\{e\}, \{h\}, \{f, g\}\big\}$ and $\mathcal{K} = \{g, h\}$. Then $\tau_{\mathcal{R}}(\mathcal{K}) = \{\emptyset, \{h\}, \{f, g\}, \{f, g, h\}, \mathcal{U}\}$. Here $\{f, g, h\}$ is $\mathcal{N}_{\alpha}\mathcal{MC}\mathfrak{cs}$ but not $\mathcal{N}_{\alpha}gb\mathfrak{cs}$.

Theorem 3.30 Every $\mathcal{N}_{\alpha}g^{*}b\mathfrak{cs}$ is $\mathcal{N}_{\alpha}\mathcal{MC}\mathfrak{cs}$.

Proof: Consider an $\mathcal{N}_{\alpha}g^{*}b\mathfrak{cs}$ $\mathcal{P}$ in $\mathcal{N}_{\alpha}\mathcal{T}$ space $\big(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{K})\big)$. For a $\mathcal{N}_{\alpha}\mathfrak{os}$ $\mathcal{S}$ in $\big(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{K})\big)$ with $\mathcal{P} \subset \mathcal{S}$, $\mathcal{N}_{\alpha}int(\mathcal{S}) = \mathcal{S}$. By Remark 2.7(i), $\mathcal{S}$ is $\mathcal{N}_{\alpha}g^{*}\mathfrak{os}$. By Remark 2.7(ii), $\mathcal{S}$ is $\mathcal{N}_{\alpha}g\mathfrak{os}$. Since $\mathcal{P}$ is $\mathcal{N}_{\alpha}g^{*}b\mathfrak{cs}$, $\mathcal{N}_{\alpha}bcl(\mathcal{P}) \subset \mathcal{S}$ whenever $\mathcal{P} \subset \mathcal{S}$ and $\mathcal{S}$ is $\mathcal{N}_{\alpha}g\mathfrak{os}$. Then $\mathcal{N}_{\alpha}bcl(\mathcal{P}) \subset \mathcal{S} = \mathcal{N}_{\alpha}int(\mathcal{S}) \subset \mathcal{N}_{\alpha}int^{*}(\mathcal{S})$. Therefore $\mathcal{N}_{\alpha}bcl(\mathcal{P}) \subset \mathcal{N}_{\alpha}int^{*}(\mathcal{S})$ whenever $\mathcal{P} \subset \mathcal{S}$ and $\mathcal{S}$ is $\mathcal{N}_{\alpha}g^{*}\mathfrak{os}$ in $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{K}))$. Thus $\mathcal{P}$ is $\mathcal{N}_{\alpha}\mathcal{MC}\mathfrak{cs}$.

Remark 3.31 Although the Theorem 3.30 holds, its converse fails in general, as shown by counterexample 3.32.

Example 3.32 Let $\mathcal{U} = \{e, f, g, h\}$ with $\mathcal{U}/\mathcal{R} = \{\{e\}, \{g\}, \{f, h\}\}$ and $\mathcal{K} = \{e\}$. Then $\tau_{\mathcal{R}}(\mathcal{K}) = \{\emptyset, \{e\}, \mathcal{U}\}$. Here $\{e, h\}$ is $\mathcal{N}_{\alpha}\mathcal{MC}\mathfrak{cs}$ but not $\mathcal{N}_{\alpha}g^{*}b\mathfrak{cs}$.

Remark 3.33 In a $\mathcal{N}_{\alpha}\mathcal{T}$ space $\big(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{K})\big)$, $\mathcal{N}_{\alpha}\beta\mathfrak{cs}$ and $\mathcal{N}_{\alpha}\mathcal{MC}\mathfrak{cs}$ are independent of each other.

Example 3.34

(i) Let $\mathcal{U} = \{e, f, g, h\}$ with $\mathcal{U}/\mathcal{R} = \big\{\{e\}, \{g\}, \{f, h\}\big\}$ and $\mathcal{K} = \{e, h\}$. Then $\tau_{\mathcal{R}}(\mathcal{K}) = \{\emptyset, \{e\}, \{f, h\}, \{e, f, h\}, \mathcal{U}\}$. Here $\{e, f\}$ is $\mathcal{N}_{\alpha}\beta\mathfrak{cs}$ but not $\mathcal{N}_{\alpha}\mathcal{MC}\mathfrak{cs}$.

(ii) Let $\mathcal{U} = \{e, f, g, h\}$ with $\mathcal{U}/\mathcal{R} = \{\{e, f\}, \{g\}, \{h\}\}$ and $\mathcal{K} = \{f, g\}$. Then $\tau_{\mathcal{R}}(\mathcal{K}) = \{\emptyset, \{g\}, \{e, f\}, \{e, f, g\}, \mathbb{U}\}$. Here $\{g\}$ is $\mathcal{N}_{\alpha}\mathcal{MC}\mathfrak{cs}$ but not $\mathcal{N}_{\alpha}\beta\mathfrak{cs}$.

Remark 3.35 The following Figure 1 illustrates the relationship between the $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}\mathfrak{cs}$ and other existing $\mathcal{N}_{\mathfrak{a}}\mathfrak{cs}$. In Figure 1,

$\mathcal{E} \rightarrow \mathcal{F}$ indicates that set $\mathcal{E}$ implies $\mathcal{F}$ but not conversely

$\mathcal{E} \nleftrightarrow \mathcal{F}$ indicates that set $\mathcal{E}$ and $\mathcal{F}$ are independent of each other.

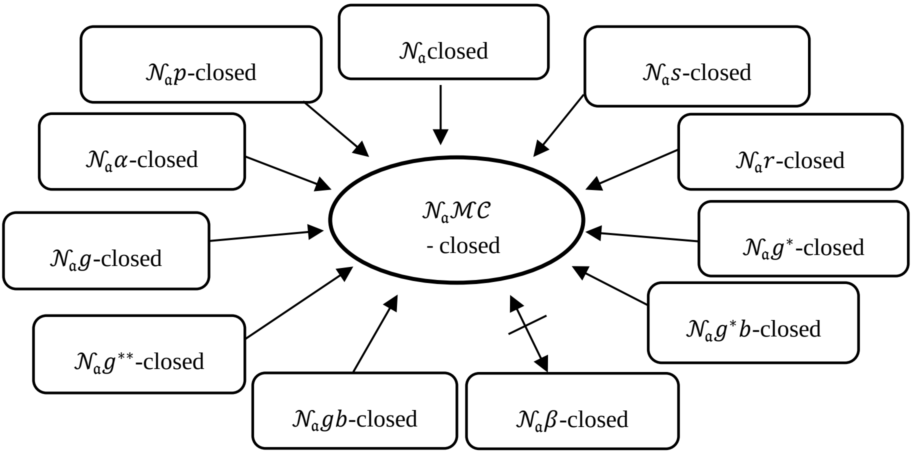

Figure 1: Implications

4. $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}$ Open Sets

Definition 4.1 Let $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{K}))$ be a $\mathcal{N}_{\mathfrak{a}}\mathcal{T}$ space. A subset $\mathcal{P}$ of a $\mathcal{N}_{\mathfrak{a}}\mathcal{T}$ space $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{K}))$ is called $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}$ open set (simply, $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}\mathfrak{os}$) if its complement is $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}$ closed in $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{K}))$.

Example 4.2 Let $\mathcal{U} = \{e, f, g, h\}$ with $\mathcal{U}/\mathcal{R} = \big\{\{e\}, \{g\}, \{f, h\}\big\}$ and $\mathcal{K} = \{e, h\}$. Then $\tau_{\mathcal{R}}(\mathcal{K}) = \{\emptyset, \{e\}, \{f, h\}, \{e, f, h\}, \mathcal{U}\}$. Here $\emptyset, \{e\}, \{f\}, \{g\}, \{h\}, \{e, g\}, \{f, g\}, \{f, h\}, \{g, h\}, \{e, f, g\}, \{e, g, h\}, \{f, g, h\}, \mathcal{U}$ are $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}$-closed sets. The $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}$-open sets are $\emptyset, \{e\}, \{f\}, \{h\}, \{e, f\}, \{e, g\}, \{e, h\}, \{f, h\}, \{e, f, g\}, \{e, f, h\}, \{e, g, h\}, \{f, g, h\}, \mathcal{U}$.

Theorem 4.3 Let $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{K}))$ be a $\mathcal{N}_{\mathfrak{a}}\mathcal{T}$ space. Then the following assertions hold:

(i) Every $\mathcal{N}_{\mathfrak{a}}\mathfrak{os}$ is $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}\mathfrak{os}$.

(ii) Every $\mathcal{N}_{\mathfrak{a}}p\mathfrak{os}$ is $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}\mathfrak{os}$.

(iii) Every $\mathcal{N}_{\mathfrak{a}}s\mathfrak{os}$ is $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}\mathfrak{os}$.

(iv) Every $\mathcal{N}_{\mathfrak{a}}\alpha\mathfrak{os}$ is $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}\mathfrak{os}$.

(v) Every $\mathcal{N}_{\mathfrak{a}}r\mathfrak{os}$ is $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}\mathfrak{os}$.

(vi) Every $\mathcal{N}_{\mathfrak{a}}g\mathfrak{os}$ is $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}\mathfrak{os}$.

(vii) Every $\mathcal{N}_{\mathfrak{a}}g^{**}\mathfrak{os}$ is $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}\mathfrak{os}$.

(viii) Every $\mathcal{N}_{\mathfrak{a}}g^{*}\mathfrak{os}$ is $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}\mathfrak{os}$.

(ix) Every $\mathcal{N}_{\mathfrak{a}}gb\mathfrak{os}$ is $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}\mathfrak{os}$.

(x) Every $\mathcal{N}_{\mathfrak{a}}g^{*}b\mathfrak{os}$ is $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}\mathfrak{os}$.

Proof: The proof is an immediate consequence of theorems 3.3, 3.6, 3.9, 3.12, 3.15, 3.18, 3.21, 3.24, 3.27, 3.30.

5. $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}$-Interior

Definition 5.1 Let $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{K}))$ be a $\mathcal{N}_{\mathfrak{a}}\mathcal{T}$ space and let $u \in \mathcal{U}$. A subset $\mathcal{P}$ of $\mathcal{U}$ is said to be $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}$-neighbourhood (briefly $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}$-nbd) of u if there exists a $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}\mathfrak{os}$ $\mathcal{F}$ containing u such that $u \in \mathcal{F} \subseteq \mathcal{P}$.

Definition 5.2 Let $\mathcal{P}$ be a subset of a $\mathcal{N}_{\mathfrak{a}}\mathcal{T}$ space $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{K}))$. A point $u \in \mathcal{P}$ is said to be $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}$-interior point of $\mathcal{P}$ if $\mathcal{P}$ is a $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}$-neighbourhood of u. The collection of all $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}$-interior points of $\mathcal{P}$ is called the $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}$-interior of $\mathcal{P}$ and is denoted by $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}^{int}(\mathcal{P})$.

Theorem 5.3 If $\mathcal{P}$ be a subset of $\big(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{K})\big)$, then $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}^{int}(\mathcal{P}) = \cup\, \{\mathcal{F} : \mathcal{F}$ is $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}\mathfrak{os}$, $\mathcal{F} \subseteq \mathcal{P}\}$.

Proof: Let $\mathcal{P}$ be a subset of $\big(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{K})\big)$. Suppose, $u \in \mathcal{N}_{\mathfrak{a}}\mathcal{MC}^{int}(\mathcal{P})$ if and only if u is a $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}$-interior point of $\mathcal{P}$ if and only if $\mathcal{P}$ is a $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}$-neighbourhood of u if and only if there exists a $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}\mathfrak{os}$ $\mathcal{F}$ containing u such that $u \in \mathcal{F} \subseteq \mathcal{P}$ if and only if $u \in \cup\, \{\mathcal{F} : \mathcal{F}$ is $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}\mathfrak{os}$, $\mathcal{F} \subseteq \mathcal{P}\}$. Hence $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}^{int}(\mathcal{P}) = \cup\, \{\mathcal{F} : \mathcal{F}$ is $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}\mathfrak{os}$, $\mathcal{F} \subseteq \mathcal{P}\}$.

Theorem 5.4 Let $\mathcal{P}$ and $\mathcal{Q}$ be subsets of $\big(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{K})\big)$. Then the following assertions hold:

(i) $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}^{int}(\mathcal{U}) = \mathcal{U}$ and $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}^{int}(\emptyset) = \emptyset$.

(ii) $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}^{int}(\mathcal{P}) \subseteq \mathcal{P}$.

(iii) If $\mathcal{Q}$ is any $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}\mathfrak{os}$ contained in $\mathcal{P}$, then $\mathcal{Q} \subseteq \mathcal{N}_{\mathfrak{a}}\mathcal{MC}^{int}(\mathcal{P})$.

(iv) If $\mathcal{P} \subseteq \mathcal{Q}$, then $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}^{int}(\mathcal{P}) \subseteq \mathcal{N}_{\mathfrak{a}}\mathcal{MC}^{int}(\mathcal{Q})$.

Proof:

(i) Since $\mathcal{U}$ and $\emptyset$ are $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}\mathfrak{os}$, by Theorem 5.3, $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}^{int}(\mathcal{U}) = \cup\, \{\mathcal{F} : \mathcal{F}$ is $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}\mathfrak{os}$, $\mathcal{F} \subseteq \mathcal{U}\} = \mathcal{U} \cup \{$ all $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}\mathfrak{os}\} = \mathcal{U}$. That is

$\mathcal{N}_a\mathcal{MC}^{int}(\mathcal{U}) = \mathcal{U}$. Since $\emptyset$ is the only $\mathcal{N}_a\mathcal{MC}\mathfrak{os}$ contained in $\emptyset$, $\mathcal{N}_a\mathcal{MC}^{int}(\emptyset) = \emptyset$.

(ii) Let $u \in \mathcal{N}_a\mathcal{MC}^{int}(\mathcal{P})$. Then u is a $\mathcal{N}_a\mathcal{MC}$-interior point of $\mathcal{P}$ that implies $\mathcal{P}$ is a $\mathcal{N}_a\mathcal{MC}$-neighbourhood of u, which implies $u \in \mathcal{F} \subseteq \mathcal{P}$, where $\mathcal{F}$ is $\mathcal{N}_a\mathcal{MC}\mathfrak{os}$. That is $u \in \mathcal{P}$. Therefore $u \in \mathcal{N}_a\mathcal{MC}^{int}(\mathcal{P})$ implies $u \in \mathcal{P}$. Hence $\mathcal{N}_a\mathcal{MC}^{int}(\mathcal{P}) \subseteq \mathcal{P}$.

(iii) Let $\mathcal{Q}$ be any $\mathcal{N}_a\mathcal{MC}\mathfrak{os}$ such that $\mathcal{Q} \subseteq \mathcal{P}$. Let $u \in \mathcal{Q}$. Since $\mathcal{Q}$ is a $\mathcal{N}_a\mathcal{MC}\mathfrak{os}$ contained in $\mathcal{P}$. u is a $\mathcal{N}_a\mathcal{MC}$-interior point of $\mathcal{P}$. That is $u \in \mathcal{N}_a\mathcal{MC}^{int}(\mathcal{P})$. Hence $\mathcal{Q} \subseteq \mathcal{N}_a\mathcal{MC}^{int}(\mathcal{P})$.

(iv) Let $\mathcal{P}$ and $\mathcal{Q}$ be subsets of $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{K}))$ such that $\mathcal{P} \subseteq \mathcal{Q}$. Let $u \in \mathcal{N}_a\mathcal{MC}^{int}(\mathcal{P})$. Then u is a $\mathcal{N}_a\mathcal{MC}$-interior point of $\mathcal{P}$ that implies $\mathcal{P}$ is a $\mathcal{N}_a\mathcal{MC}$-neighbourhood of u. Since $\mathcal{P} \subseteq \mathcal{Q}$, $\mathcal{Q}$ is also a $\mathcal{N}_a\mathcal{MC}$-neighbourhood of u. That implies $u \in \mathcal{N}_a\mathcal{MC}^{int}(\mathcal{Q})$. Hence $\mathcal{N}_a\mathcal{MC}^{int}(\mathcal{P}) \subseteq \mathcal{N}_a\mathcal{MC}^{int}(\mathcal{Q})$.

Theorem 5.5 Let $\mathcal{P}$ be a subset of a $\mathcal{N}_a\mathcal{T}$ space $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{K}))$. If $\mathcal{P}$ is $\mathcal{N}_a\mathcal{MC}\mathfrak{os}$, then $\mathcal{N}_a\mathcal{MC}^{int}(\mathcal{P}) = \mathcal{P}$.

Proof: Consider an $\mathcal{N}_a\mathcal{MC}\mathfrak{os}$ $\mathcal{P}$ in $\mathcal{N}_a\mathcal{T}$ space $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{K}))$. Then $\mathcal{N}_a\mathcal{MC}^{int}(\mathcal{P}) \subseteq \mathcal{P}$. Also $\mathcal{P}$ is $\mathcal{N}_a\mathcal{MC}\mathfrak{os}$ contained in $\mathcal{P}$. By theorem 5.4(iii), $\mathcal{P} \subseteq \mathcal{N}_a\mathcal{MC}^{int}(\mathcal{P})$. Hence $\mathcal{N}_a\mathcal{MC}^{int}(\mathcal{P}) = \mathcal{P}$.

Theorem 5.6 If $\mathcal{P}$ and $\mathcal{Q}$ are subsets of $(\mathbb{U}, \tau_{\mathbb{R}}(\mathbb{X}))$, then the following assertions hold.

(i) $\mathcal{N}_a\mathcal{MC}^{int}(\mathcal{P}) \cup \mathcal{N}_a\mathcal{MC}^{int}(\mathcal{Q}) \subseteq \mathcal{N}_a\mathcal{MC}^{int}(\mathcal{P} \cup \mathcal{Q})$.

(ii) $\mathcal{N}_a\mathcal{MC}^{int}(\mathcal{P} \cap \mathcal{Q}) \subseteq \mathcal{N}_a\mathcal{MC}^{int}(\mathcal{P}) \cap \mathcal{N}_a\mathcal{MC}^{int}(\mathcal{Q})$.

Proof:

(i) Since $\mathcal{P} \subseteq \mathcal{P} \cup \mathcal{Q}$ and $\mathcal{Q} \subseteq \mathcal{P} \cup \mathcal{Q}$. By Theorem 5.4$(iv)$, $\mathcal{N}_a\mathcal{MC}^{int}(\mathcal{P}) \subseteq \mathcal{N}_a\mathcal{MC}^{int}(\mathcal{P} \cup \mathcal{Q})$ and $\mathcal{N}_a\mathcal{MC}^{int}(\mathcal{Q}) \subseteq \mathcal{N}_a\mathcal{MC}^{int}(\mathcal{P} \cup \mathcal{Q})$. Therefore $\mathcal{N}_a\mathcal{MC}^{int}(\mathcal{P}) \cup \mathcal{N}_a\mathcal{MC}^{int}(\mathcal{Q}) \subseteq \mathcal{N}_a\mathcal{MC}^{int}(\mathcal{P} \cup \mathcal{Q})$.

(ii) Since $\mathcal{P} \cap \mathcal{Q} \subseteq \mathcal{P}$ and $\mathcal{P} \cap \mathcal{Q} \subseteq \mathcal{Q}$, from Theorem 5.4$(iv)$,$\mathcal{N}_a\mathcal{MC}^{int}(\mathcal{P} \cap \mathcal{Q}) \subseteq \mathcal{N}_a\mathcal{MC}^{int}(\mathcal{P})$ and $\mathcal{N}_a\mathcal{MC}^{int}(\mathcal{P} \cap \mathcal{Q}) \subseteq \mathcal{N}_a\mathcal{MC}^{int}(\mathcal{Q})$. Therefore $\mathcal{N}_a\mathcal{MC}^{int}(\mathcal{P} \cap \mathcal{Q}) \subseteq \mathcal{N}_a\mathcal{MC}^{int}(\mathcal{P}) \cap \mathcal{N}_a\mathcal{MC}^{int}(\mathcal{Q})$.

Remark 5.7 Equality does not hold for the theorem 5.6 as illustrated by the example 5.8.

Example 5.8

(i) Let $\mathcal{U} = \{e, f, g, h\}$ with $\mathcal{U}/\mathcal{R} = \{\{e\}, \{g\}, \{f, h\}\}$ and $\mathcal{K} = \{e, h\}$. Then $\tau_{\mathcal{R}}(\mathcal{K}) = \{\emptyset, \{e\}, \{f, h\}, \{e, f, h\}, \mathcal{U}\}$. Let $\mathcal{P} = \{g\}$ and $\mathcal{Q} = \{e, h\}$.

Then $\mathcal{P} \cup \mathcal{Q} = \{e, g, h\}$. Also $\mathcal{N}_\alpha\mathcal{MC}^{int}(\mathcal{P}) = \emptyset$ and $\mathcal{N}_\alpha\mathcal{MC}^{int}(\mathcal{Q}) = \{e, h\}$ and $\mathcal{N}_\alpha\mathcal{MC}^{int}(\mathcal{P} \cup \mathcal{Q}) = \{e, g, h\}$. Hence $\mathcal{N}_\alpha\mathcal{MC}^{int}(\mathcal{P}) \cup \mathcal{N}_\alpha\mathcal{MC}^{int}(\mathcal{Q}) \neq \mathcal{N}_\alpha\mathcal{MC}^{int}(\mathcal{P} \cup \mathcal{Q})$.

(ii) Let $\mathcal{U} = \{e, f, g, h\}$ with $\mathcal{U}/\mathcal{R} = \{\{e\}, \{g\}, \{f, h\}\}$ and $\mathcal{K} = \{e, h\}$. Then $\tau_{\mathcal{R}}(\mathcal{K}) = \{\emptyset, \{e\}, \{f, h\}, \{e, f, h\}, \mathcal{U}\}$. Let $\mathcal{P} = \{e, g\}$ and $\mathcal{Q} = \{e, f, h\}$. Then $\mathcal{P} \cap \mathcal{Q} = \{e\}$. Also $\mathcal{N}_\alpha\mathcal{MC}^{int}(\mathcal{P}) = \{e, g\}$, $\mathcal{N}_\alpha\mathcal{MC}^{int}(\mathcal{Q}) = \mathcal{U}$ and $\mathcal{N}_\alpha\mathcal{MC}^{int}(\mathcal{P} \cap \mathcal{Q}) = \{e\}$. Hence $\mathcal{N}_\alpha\mathcal{MC}^{int}(\mathcal{P}) \cap \mathcal{N}_\alpha\mathcal{MC}^{int}(\mathcal{Q}) \neq \mathcal{N}_\alpha\mathcal{MC}^{int}(\mathcal{P} \cap \mathcal{Q})$.

Theorem 5.9 Let $\mathcal{P}$ be a subset of a $\mathcal{N}_\alpha\mathcal{T}$ space $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{K}))$. Then $\mathcal{N}_\alpha int(\mathcal{P}) \subseteq \mathcal{N}_\alpha\mathcal{MC}^{int}(\mathcal{P})$.

Proof: Consider an $\mathcal{N}_\alpha\mathfrak{os}$ $\mathcal{P}$ in $\mathcal{N}_\alpha\mathcal{T}$ space $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{K}))$. Let $u \in \mathcal{N}_\alpha int(\mathcal{P})$. Then $u \in \cup \{\mathcal{F} : \mathcal{F}$ is $\mathcal{N}_\alpha\mathfrak{os}$, $\mathcal{F} \subseteq \mathcal{P}\}$ implies there exists a $\mathcal{N}_\alpha\mathfrak{os}$ $\mathcal{F}$ such that $u \in \mathcal{F} \subseteq \mathcal{P}$ implies there exists a $\mathcal{N}_\alpha\mathcal{MC}\mathfrak{os}$ $\mathcal{F}$ such that $u \in \mathcal{F} \subseteq \mathcal{P}$, since every $\mathcal{N}_\alpha\mathfrak{os}$ is a $\mathcal{N}_\alpha\mathcal{MC}\mathfrak{os}$ in $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{K}))$ implies $u \in \cup \{\mathcal{F} : \mathcal{F}$ is $\mathcal{N}_\alpha\mathcal{MC}\mathfrak{os}$, $\mathcal{F} \subseteq \mathcal{P}\}$ implies $u \in \mathcal{N}_\alpha\mathcal{MC}^{int}(\mathcal{P})$. Thus $u \in \mathcal{N}_\alpha int(\mathcal{P})$ implies $u \in \mathcal{N}_\alpha\mathcal{MC}^{int}(\mathcal{P})$. Hence$\mathcal{N}_\alpha int(\mathcal{P}) \subseteq \mathcal{N}_\alpha\mathcal{MC}^{int}(\mathcal{P})$.

Remark 5.10 Equality does not hold for the theorem 5.9 as illustrated by the example 5.11.

Example 5.11 Let $\mathcal{U} = \{e, f, g, h\}$ with $\mathcal{U}/\mathcal{R} = \{\{e, f\}, \{g\}, \{h\}\}$ and $\mathcal{K} = \{f, g\}$. Then $\tau_{\mathcal{R}}(\mathcal{K}) = \{\emptyset, \{g\}, \{e, f\}, \{e, f, g\}, \mathbb{U}\}$. Let $\mathcal{P} = \{e, f, h\}$. Then $\mathcal{N}_\alpha int(\mathcal{P}) = \{e, f\}$ and $\mathcal{N}_\alpha\mathcal{MC}^{int}(\mathcal{P}) = \{e, f, h\}$. Hence $\mathcal{N}_\alpha int(\mathcal{P}) \neq \mathcal{N}_\alpha\mathcal{MC}^{int}(\mathcal{P})$.

6. $\mathcal{N}_\alpha\mathcal{MC}$-Closure

Definition 6.1 Let $\mathcal{P}$ be a subset of a $\mathcal{N}_\alpha\mathcal{T}$ space $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{K}))$. The $\mathcal{N}_\alpha\mathcal{MC}$-closure of $\mathcal{P}$ (simply, $\mathcal{N}_\alpha\mathcal{MC}^{cl}(\mathcal{P})$) is the intersection of all $\mathcal{N}_\alpha\mathcal{MC}\mathfrak{cs}$ containing $\mathcal{P}$.

$$\mathcal{N}_\alpha\mathcal{MC}^{cl}(\mathcal{P}) = \cap \{\mathcal{H} : \mathcal{H} \text{ is } \mathcal{N}_\alpha\mathcal{MC}\mathfrak{cs}, \mathcal{P} \subseteq \mathcal{H}\}.$$

Theorem 6.2 Let $\mathcal{P}$ and $\mathcal{Q}$ be subsets of $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{K}))$. Then the following assertions hold:

(i) $\mathcal{N}_\alpha\mathcal{MC}^{clt}(\mathcal{U}) = \mathcal{U}$ and $\mathcal{N}_\alpha\mathcal{MC}^{cl}(\emptyset) = \emptyset$.

(ii) $\mathcal{P} \subseteq \mathcal{N}_\alpha\mathcal{MC}^{cl}(\mathcal{P})$.

(iii) If $\mathcal{Q}$ is any $\mathcal{N}_\alpha\mathcal{MC}\mathfrak{cs}$ containing $\mathcal{P}$, then $\mathcal{N}_\alpha\mathcal{MC}^{cl}(\mathcal{P}) \subseteq \mathcal{Q}$.

(iv) If $\mathcal{P} \subseteq \mathcal{Q}$, then $\mathcal{N}_\alpha\mathcal{MC}^{cl}(\mathcal{P}) \subseteq \mathcal{N}_\alpha\mathcal{MC}^{cl}(\mathcal{Q})$.

Proof: Follows from the definition of $\mathcal{N}_\alpha\mathcal{MC}$-closure.

Theorem 6.3 Let $\mathcal{P}$ be a subset of a $\mathcal{N}_\alpha\mathcal{T}$ space $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{K}))$. If $\mathcal{P}$ is $\mathcal{N}_\alpha\mathcal{MC}\mathfrak{cs}$, then $\mathcal{N}_\alpha\mathcal{MC}^{cl}(\mathcal{P}) = \mathcal{P}$.

Proof: Consider an $\mathcal{N}_{\alpha}\mathcal{MC}$cs $\mathcal{P}$ in $\mathcal{N}_{\alpha}\mathcal{T}$ space $(\mathcal{U},\tau_{\mathcal{R}}(\mathcal{K}))$. Then $\mathcal{P}\subseteq\mathcal{N}_{\alpha}\mathcal{MC}^{cl}(\mathcal{P})$. Also $\mathcal{P}$ is $\mathcal{N}_{\alpha}\mathcal{MC}$cs containing $\mathcal{P}$. By theorem 6.2 (iii), $\mathcal{N}_{\alpha}\mathcal{MC}^{cl}(\mathcal{P})\subseteq\mathcal{P}$. Hence $\mathcal{N}_{\alpha}\mathcal{MC}^{cl}(\mathcal{P})=\mathcal{P}$.

Theorem 6.4 If $\mathcal{P}$ and $\mathcal{Q}$ are subsets of $(\mathcal{U},\tau_{\mathcal{R}}(\mathcal{K}))$, then the following assertions hold.

(i) $\mathcal{N}_{\alpha}\mathcal{MC}^{cl}(\mathcal{P})\cup\mathcal{N}_{\alpha}\mathcal{MC}^{cl}(\mathcal{Q})\subseteq\mathcal{N}_{\alpha}\mathcal{MC}^{cl}(\mathcal{P}\cup\mathcal{Q})$.

(ii) $\mathcal{N}_{\alpha}\mathcal{MC}^{cl}(\mathcal{P}\cap\mathcal{Q})\subseteq\mathcal{N}_{\alpha}\mathcal{MC}^{cl}(\mathcal{P})\cap\mathcal{N}_{\alpha}\mathcal{MC}^{cl}(\mathcal{Q})$.

Proof: (i) Let $\mathcal{P}$ and $\mathcal{Q}$ are subsets of $\mathcal{U}$. Clearly $\mathcal{P}\subseteq\mathcal{P}\cup\mathcal{Q}$ and $\mathcal{Q}\subseteq\mathcal{P}\cup\mathcal{Q}$. By Theorem 6.2 (iv), $\mathcal{N}_{\alpha}\mathcal{MC}^{cl}(\mathcal{P})\subseteq\mathcal{N}_{\alpha}\mathcal{MC}^{cl}(\mathcal{P}\cup\mathcal{Q})$ and $\mathcal{N}_{\alpha}\mathcal{MC}^{cl}(\mathcal{Q})\subseteq\mathcal{N}_{\alpha}\mathcal{MC}^{cl}(\mathcal{P}\cup\mathcal{Q})$.

Therefore $\mathcal{N}_{\alpha}\mathcal{MC}^{cl}(\mathcal{P})\cup\mathcal{N}_{\alpha}\mathcal{MC}^{cl}(\mathcal{Q})\subseteq\mathcal{N}_{\alpha}\mathcal{MC}^{cl}(\mathcal{P}\cup\mathcal{Q})$.

(ii) Let $\mathcal{P}$ and $\mathcal{Q}$ are subsets of $\mathcal{U}$. Clearly $\mathcal{P}\cap\mathcal{Q}\subseteq\mathcal{P}$ and $\mathcal{P}\cap\mathcal{Q}\subseteq\mathcal{Q}$. By Theorem 6.2 *(iv)*, $\mathcal{N}_{\alpha}\mathcal{MC}^{cl}(\mathcal{P}\cap\mathcal{Q})\subseteq\mathcal{N}_{\alpha}\mathcal{MC}^{cl}(\mathcal{P})$ and $\mathcal{N}_{\alpha}\mathcal{MC}^{cl}(\mathcal{P}\cap\mathcal{Q})\subseteq\mathcal{N}_{\alpha}\mathcal{MC}^{cl}(\mathcal{Q})$.

Therefore $\mathcal{N}_{\alpha}\mathcal{MC}^{cl}(\mathcal{P}\cap\mathcal{Q})\subseteq\mathcal{N}_{\alpha}\mathcal{MC}^{cl}(\mathcal{P})\cap\mathcal{N}_{\alpha}\mathcal{MC}^{cl}(\mathcal{Q})$.

Remark 6.5 In theorem 6.4, equality does not hold as illustrated by the example 6.6.

Example 6.6 (i) Let $\mathcal{U}=\{e,f,g,h\}$ with $\mathcal{U}/\mathcal{R}=\{\{e\},\{g\},\{f,h\}\}$ and $\mathcal{K}=\{e,h\}$. Then $\tau_{\mathcal{R}}(\mathcal{K})=\{\emptyset,\{e\},\{f,h\},\{e,f,h\},\mathcal{U}\}$. Let $\mathcal{P}=\{e\}$ and $\mathcal{Q}=\{f,h\}$. Then $\mathcal{P}\cup\mathcal{Q}=\{e,f,h\}$. Also $\mathcal{N}_{\alpha}\mathcal{MC}^{cl}(\mathcal{P})=\{e\}$ and $\mathcal{N}_{\alpha}\mathcal{MC}^{cl}(\mathcal{Q})=\{f,h\}$ and $\mathcal{N}_{\alpha}\mathcal{MC}^{cl}(\mathcal{P}\cup\mathcal{Q})=\mathcal{U}$. Hence $\mathcal{N}_{\alpha}\mathcal{MC}^{cl}(\mathcal{P})\cup\mathcal{N}_{\alpha}\mathcal{MC}^{cl}(\mathcal{Q})\neq\mathcal{N}_{\alpha}\mathcal{MC}^{cl}(\mathcal{P}\cup\mathcal{Q})$.

(ii) Let $\mathcal{U}=\{e,f,g,h\}$ with $\mathcal{U}/\mathcal{R}=\{\{e\},\{g\},\{f,h\}\}$ and $\mathcal{K}=\{e,h\}$. Then $\tau_{\mathcal{R}}(\mathcal{K})=\{\emptyset,\{e\},\{f,h\},\{e,f,h\},\mathcal{U}\}$. Let $\mathcal{P}=\{f,g\}$ and $\mathcal{Q}=\{e,f,h\}$. Then $\mathcal{P}\cap\mathcal{Q}=\{f\}$. Also $\mathcal{N}_{\alpha}\mathcal{MC}^{cl}(\mathcal{P})=\{f,g\}$, $\mathcal{N}_{\alpha}\mathcal{MC}^{cl}(\mathcal{Q})=\mathcal{U}$ and $\mathcal{N}_{\alpha}\mathcal{MC}^{cl}(\mathcal{P}\cap\mathcal{Q})=\{f\}$.

Hence $\mathcal{N}_{\alpha}\mathcal{MC}^{cl}(\mathcal{P})\cap\mathcal{N}_{\alpha}\mathcal{MC}^{cl}(\mathcal{Q})\neq\mathcal{N}_{\alpha}\mathcal{MC}^{cl}(\mathcal{P}\cap\mathcal{Q})$.

Theorem 6.7 Let $\mathcal{P}$ be a subset of a $\mathcal{N}_{\alpha}\mathcal{T}$ space $(\mathcal{U},\tau_{\mathcal{R}}(\mathcal{K}))$. For any $u\in\mathcal{U}$, $u\in\mathcal{N}_{\alpha}\mathcal{MC}^{cl}(\mathcal{P})$ if and only if $\mathcal{H}\cap\mathcal{P}\neq\emptyset$ for every $\mathcal{N}_{\alpha}\mathcal{MC}$os $\mathcal{H}$ containing u.

Proof: Necessity: Let $u\in\mathcal{U}$. Suppose $u\in\mathcal{N}_{\alpha}\mathcal{MC}^{cl}(\mathcal{P})$. To prove $\mathcal{H}\cap\mathcal{P}\neq\emptyset$ for every $\mathcal{N}_{\alpha}\mathcal{MC}$os $\mathcal{H}$ containing u. Suppose there exists a $\mathcal{N}_{\alpha}\mathcal{MC}$os $\mathcal{H}$ containing u such that $\mathcal{H}\cap\mathcal{P}=\emptyset$. Then $\mathcal{P}\subseteq\mathcal{U}\backslash\mathcal{H}$ and $\mathcal{U}\backslash\mathcal{H}$ is $\mathcal{N}_{\alpha}\mathcal{MC}$cs. By theorem 6.2$(iii)$, $\mathcal{N}_{\alpha}\mathcal{MC}^{cl}(\mathcal{P})\subseteq\mathcal{U}\backslash\mathcal{H}$. Thus $u\notin\mathcal{N}_{\alpha}\mathcal{MC}^{cl}(\mathcal{P})$, which is a contradiction. Hence $\mathcal{H}\cap\mathcal{P}\neq\emptyset$ for every $\mathcal{N}_{\alpha}\mathcal{MC}$os $\mathcal{H}$ containing u.

Sufficiency: Let $\mathcal{H} \cap \mathcal{P} \neq \emptyset$ for every $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}\mathfrak{os}$ $\mathcal{H}$ containing u. To prove $u \in \mathcal{N}_{\mathfrak{a}}\mathcal{MC}^{cl}(\mathcal{P})$. Suppose $u \notin \mathcal{N}_{\mathfrak{a}}\mathcal{MC}^{cl}(\mathcal{P})$. Then there exists a $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}\mathfrak{cs}$ set $\mathcal{W}$ such that $\mathcal{P} \subseteq \mathcal{W}$ and $u \in \mathcal{U}\backslash\mathcal{W}$. Therefore $\mathcal{P} \cap (\mathcal{U}\backslash\mathcal{W}) = \emptyset$ and $\mathcal{U}\backslash\mathcal{W}$ is $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}\mathfrak{os}$ which is a contradiction to the hypothesis. Hence $u \in \mathcal{N}_{\mathfrak{a}}\mathcal{MC}^{cl}(\mathcal{P})$.

Theorem 6.8 Let $\mathcal{P}$ be a subset of a $\mathcal{N}_{\mathfrak{a}}\mathcal{T}$ space $\big(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{K})\big)$. Then the following assertions hold:

(i) $\mathcal{U} \setminus \mathcal{N}_{\mathfrak{a}}\mathcal{MC}^{int}(\mathcal{P}) = \mathcal{N}_{\mathfrak{a}}\mathcal{MC}^{cl}(\mathcal{U} \setminus \mathcal{P})$.

(ii) $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}^{int}(\mathcal{P}) = \mathcal{U} \setminus \mathcal{N}_{\mathfrak{a}}\mathcal{MC}^{cl}(\mathcal{U} \setminus \mathcal{P})$.

Proof: (i) Necessity: Let $u \in \mathcal{U} \setminus \mathcal{N}_{\mathfrak{a}}\mathcal{MC}^{int}(\mathcal{P})$. This implies $u \notin \mathcal{N}_{\mathfrak{a}}\mathcal{MC}^{int}(\mathcal{P})$. That is every $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}\mathfrak{os}$ $\mathcal{H}$ containing u such that $u \in \mathcal{H} \nsubseteq \mathcal{P}$. That is every $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}\mathfrak{os}$ $\mathcal{H}$ containing u such that $\mathcal{H} \cap (\mathcal{U}\backslash\mathcal{P}) \neq \emptyset$. By Theorem 6.7, $u \in \mathcal{N}_{\mathfrak{a}}\mathcal{MC}^{cl}(\mathcal{U} \setminus \mathcal{P})$. Hence $\mathcal{U} \setminus \mathcal{N}_{\mathfrak{a}}\mathcal{MC}^{int}(\mathcal{P}) \subseteq \mathcal{N}_{\mathfrak{a}}\mathcal{MC}^{cl}(\mathcal{U} \setminus \mathcal{P})$.

Sufficiency: Let $u \in \mathcal{N}_{\mathfrak{a}}\mathcal{MC}^{cl}(\mathcal{U} \setminus \mathcal{P})$. Then by theorem 6.7 every $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}\mathfrak{os}$ $\mathcal{H}$ containing u such that $\mathcal{H} \cap (\mathcal{U}\backslash\mathcal{P}) \neq \emptyset$. That is every $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}\mathfrak{os}$ $\mathcal{H}$ containing u such that $u \in \mathcal{H} \nsubseteq \mathcal{P}$. This implies by definition 5.1, $u \notin \mathcal{N}_{\mathfrak{a}}\mathcal{MC}^{int}(\mathcal{P})$. That is $u \in \mathcal{U}\backslash\mathcal{N}_{\mathfrak{a}}\mathcal{MC}^{int}(\mathcal{P})$. Hence $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}^{cl}(\mathcal{U} \setminus \mathcal{P}) \subseteq \mathcal{U} \setminus \mathcal{N}_{\mathfrak{a}}\mathcal{MC}^{int}(\mathcal{P})$. Therefore $\mathcal{U} \setminus \mathcal{N}_{\mathfrak{a}}\mathcal{MC}^{int}(\mathcal{P}) = \mathcal{N}_{\mathfrak{a}}\mathcal{MC}^{cl}(\mathcal{U} \setminus \mathcal{P})$.

(ii) Taking complement of (i), it follows that $\mathcal{N}_{\mathfrak{a}}\mathcal{MC}^{int}(\mathcal{P}) = \mathcal{U} \setminus \mathcal{N}_{\mathfrak{a}}\mathcal{MC}^{cl}(\mathcal{U} \setminus \mathcal{P})$.

7. Attribute Reduction in Student Dropout Risk Using Nano Topological Bases

Student dropout is one of the major challenges faced by higher education institutions. Various academic, financial, personal, and social factors may influence a student's decision to discontinue their studies. Identifying the factors associated with dropout risk is important for improving student retention and supporting effective educational planning.

In this study, a survey was conducted among eight students belonging to Undergraduate (UG), Postgraduate (PG), and Doctoral programmes. Information related to factors influencing dropout risk was collected and analyzed. Using nano topological concepts, the significant attributes affecting dropout risk are identified, and redundant attributes are removed through attribute reduction. The obtained results provide a useful framework for understanding dropout risk and supporting decision-making in higher education institutions.

Table 1

S.NO	Academic Performance	Student Engagement	Travel Distance	Health and Family	Financial Impact	Dropout Risk
$\mathcal{B}_1$	Satisfactory	Yes	Yes	Yes	No	No
$\mathcal{B}_2$	Satisfactory	Yes	Yes	Yes	No	No
$\mathcal{B}_3$	Unsatisfactory	Yes	No	Yes	Yes	Yes
$\mathcal{B}_4$	Satisfactory	No	Yes	No	Yes	Yes
$\mathcal{B}_5$	Unsatisfactory	Yes	No	Yes	Yes	Yes
$\mathcal{B}_6$	Satisfactory	Yes	No	Yes	Yes	No
$\mathcal{B}_7$	Unsatisfactory	No	No	Yes	Yes	Yes
$\mathcal{B}_8$	Unsatisfactory	No	No	Yes	No	No

Here $\mathcal{U} = \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5, \mathcal{B}_6, \mathcal{B}_7, \mathcal{B}_8\}$, the set of students and $\mathcal{A} =$ {Academic Performance, Student Engagement, Travel Distance, Health and Family, Financial Impact, Dropout Risk} which is divided into a set $\mathcal{C}$ of condition attributes given by {AP, SE, TD, HF, FI} and a set $\mathcal{D}$ of decision attribute given by {Dropout Risk}. The family of equivalence classes, $\mathcal{U}/\mathcal{C}$ corresponding to $\mathcal{C}$ is given by

$$\mathcal{U}/\mathcal{R}(\mathcal{C}) = \{\{\mathcal{B}_1, \mathcal{B}_2\}, \{\mathcal{B}_3, \mathcal{B}_5\}, \{\mathcal{B}_4\}, \{\mathcal{B}_6\}, \{\mathcal{B}_7\}, \{\mathcal{B}_8\}\}.$$

Case 1: (Students at Risk of Dropping Out)

Let $\mathcal{K} = \{\mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5, \mathcal{B}_7\}$ be the set of students at risk of dropping out. Then the lower and upper approximations and boundary regions of $\mathcal{K}$ are $L_{\mathcal{C}}(\mathcal{K}) = \{\mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5, \mathcal{B}_7\}$, $U_{\mathcal{C}}(\mathcal{K}) = \{\mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5, \mathcal{B}_7\}$ and $B_{\mathcal{C}}(\mathcal{K}) = \emptyset$ respectively. Therefore, the nano topology is $\tau_{\mathcal{C}}(\mathcal{K}) = \{\emptyset, \{\mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5, \mathcal{B}_7\}, \mathcal{U}\}$ and its basis is $\beta_{\mathcal{C}}(\mathcal{K}) = \{\mathcal{U}, \{\mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5, \mathcal{B}_7\}, \emptyset\}$.

Step 1:

When the attribute "Academic Performance" is excluded from $\mathcal{C}$,

$\mathcal{U}/\mathcal{R}(\mathcal{C} - AP) = \{\{\mathcal{B}_1, \mathcal{B}_2\}, \{\mathcal{B}_3, \mathcal{B}_5, \mathcal{B}_6\}, \{\mathcal{B}_4\}, \{\mathcal{B}_7\}, \{\mathcal{B}_8\}\}$. Here

$L_{\mathcal{C}-\{AP\}}(\mathcal{K}) = \{\mathcal{B}_4, \mathcal{B}_7\}$, $U_{\mathcal{C}-\{AP\}}(\mathcal{K}) = \{\mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5, \mathcal{B}_6, \mathcal{B}_7\}$ and $B_{\mathcal{C}-\{AP\}}(\mathcal{K}) = \{\mathcal{B}_3, \mathcal{B}_5, \mathcal{B}_6\}$.

Therefore $\tau_{\mathcal{C}-\{AP\}}(\mathcal{K}) = \{\emptyset, \{\mathcal{B}_4, \mathcal{B}_7\}, \{\mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5, \mathcal{B}_6, \mathcal{B}_7\}, \{\mathcal{B}_3, \mathcal{B}_5, \mathcal{B}_6\}, \mathcal{U}\}$ and its basis is $\beta_{\mathcal{C}-\{AP\}}(\mathcal{K}) = \{\mathcal{U}, \{\mathcal{B}_4, \mathcal{B}_7\}, \{\mathcal{B}_3, \mathcal{B}_5, \mathcal{B}_6\}\}$.

Hence, $\beta_{\mathcal{C}-\{AP\}}(\mathcal{K}) \neq \beta_{\mathcal{C}}(\mathcal{K})$.

Step 2:

When the attribute "Student Engagement" is excluded from $\mathcal{C}$,

$\mathcal{U}/\mathcal{R}(\mathcal{C}-SE)=\{\{\mathcal{B}_1,\mathcal{B}_2\},\{\mathcal{B}_3,\mathcal{B}_5,\mathcal{B}_7\},\{\mathcal{B}_4\},\{\mathcal{B}_6\},\{\mathcal{B}_8\}\}$.

Here $L_{\mathcal{C}-\{SE\}}(\mathcal{K})=\{\mathcal{B}_3,\mathcal{B}_4,\mathcal{B}_5,\mathcal{B}_7\}$, $U_{\mathcal{C}-\{SE\}}(\mathcal{K})=\{\mathcal{B}_3,\mathcal{B}_4,\mathcal{B}_5,\mathcal{B}_7\}$ and $B_{\mathcal{C}-\{SE\}}(\mathcal{K})=\emptyset$. Therefore $\tau_{\mathcal{C}-\{SE\}}(\mathcal{K})=\{\emptyset,\{\mathcal{B}_3,\mathcal{B}_4,\mathcal{B}_5,\mathcal{B}_7\},\mathcal{U}\}$ and its basis is

$\beta_{\mathcal{C}-\{SE\}}(\mathcal{K})=\{\mathcal{U},\{\mathcal{B}_3,\mathcal{B}_4,\mathcal{B}_5,\mathcal{B}_7\},\emptyset\}$.

Hence, $\beta_{\mathcal{C}-\{SE\}}(\mathcal{K})=\beta_{\mathcal{C}}(\mathcal{K})$.

Step 3:

When the attribute "Travel Distance" is excluded from $\mathcal{C}$,

$\mathcal{U}/\mathcal{R}(\mathcal{C}-TD)=\{\{\mathcal{B}_1,\mathcal{B}_2\},\{\mathcal{B}_3,\mathcal{B}_5\},\{\mathcal{B}_4\},\{\mathcal{B}_6\},\{\mathcal{B}_7\},\{\mathcal{B}_8\}\}$.

Here $L_{\mathcal{C}-\{TD\}}(\mathcal{K})=\{\mathcal{B}_3,\mathcal{B}_4,\mathcal{B}_5,\mathcal{B}_7\}$, $U_{\mathcal{C}-\{TD\}}(\mathcal{K})=\{\mathcal{B}_3,\mathcal{B}_4,\mathcal{B}_5,\mathcal{B}_7\}$ and $B_{\mathcal{C}-\{TD\}}(\mathcal{K})=\emptyset$. Therefore $\tau_{\mathcal{C}-\{TD\}}(\mathcal{K})=\{\emptyset,\{\mathcal{B}_3,\mathcal{B}_4,\mathcal{B}_5,\mathcal{B}_7\},\mathcal{U}\}$ and its basis is

$\beta_{\mathcal{C}-\{TD\}}(\mathcal{K})=\{\mathcal{U},\{\mathcal{B}_3,\mathcal{B}_4,\mathcal{B}_5,\mathcal{B}_7\},\emptyset\}$.

Hence, $\beta_{\mathcal{C}-\{TD\}}(\mathcal{K})=\beta_{\mathcal{C}}(\mathcal{K})$.

Step 4:

When the attribute "Health and Family" is excluded from $\mathcal{C}$,

$\mathcal{U}/\mathcal{R}(\mathcal{C}-HF)=\{\{\mathcal{B}_1,\mathcal{B}_2\},\{\mathcal{B}_3,\mathcal{B}_5\},\{\mathcal{B}_4\},\{\mathcal{B}_6\},\{\mathcal{B}_7\},\{\mathcal{B}_8\}\}$.

Here $L_{\mathcal{C}-\{HF\}}(\mathcal{K})=\{\mathcal{B}_3,\mathcal{B}_4,\mathcal{B}_5,\mathcal{B}_7\}$, $U_{\mathcal{C}-\{HF\}}(\mathcal{K})=\{\mathcal{B}_3,\mathcal{B}_4,\mathcal{B}_5,\mathcal{B}_7\}$ and $B_{\mathcal{C}-\{HF\}}(\mathcal{K})=\emptyset$. Therefore $\tau_{\mathcal{C}-\{HF\}}(\mathcal{K})=\{\emptyset,\{\mathcal{B}_3,\mathcal{B}_4,\mathcal{B}_5,\mathcal{B}_7\},\mathcal{U}\}$ and its basis is

$\beta_{\mathcal{C}-\{HF\}}(\mathcal{K})=\{\mathcal{U},\{\mathcal{B}_3,\mathcal{B}_4,\mathcal{B}_5,\mathcal{B}_7\},\emptyset\}$.

Hence, $\beta_{\mathcal{C}-\{HF\}}(\mathcal{K})=\beta_{\mathcal{C}}(\mathcal{K})$.

Step 5:

When the attribute "Financial Impact" is excluded from $\mathcal{C}$,

$\mathcal{U}/\mathcal{R}(\mathcal{C}-FI)=\{\{\mathcal{B}_1,\mathcal{B}_2\},\{\mathcal{B}_3,\mathcal{B}_5\},\{\mathcal{B}_4\},\{\mathcal{B}_6\},\{\mathcal{B}_7,\mathcal{B}_8\}\}$. Here

$L_{\mathcal{C}-\{FI\}}(\mathcal{K})=\{\mathcal{B}_3,\mathcal{B}_4,\mathcal{B}_5\}$, $U_{\mathcal{C}-\{FI\}}(\mathcal{K})=\{\mathcal{B}_3,\mathcal{B}_4,\mathcal{B}_5,\mathcal{B}_7,\mathcal{B}_8\}$ and $B_{\mathcal{C}-\{FI\}}(\mathcal{K})=\{\mathcal{B}_7,\mathcal{B}_8\}$. Therefore $\tau_{\mathcal{C}-\{IF\}}(\mathcal{K})=\{\emptyset,\{\mathcal{B}_3,\mathcal{B}_4,\mathcal{B}_5\},\{\mathcal{B}_3,\mathcal{B}_4,\mathcal{B}_5,\mathcal{B}_7,\mathcal{B}_8\},\{\mathcal{B}_7,\mathcal{B}_8\},\mathcal{U}\}$ and its basis is $\beta_{\mathcal{C}-\{FI\}}(\mathcal{K})=\{\mathcal{U},\{\mathcal{B}_3,\mathcal{B}_4,\mathcal{B}_5\},\{\mathcal{B}_7,\mathcal{B}_8\}\}$.

Hence, $\beta_{\mathcal{C}-\{FI\}}(\mathcal{K})\neq\beta_{\mathcal{C}}(\mathcal{K})$.

Therefore CORE = {Academic Performance, Financial Impact}.

Case 2: (Students Not at Risk of Dropping Out)

Let $\mathcal{K} = \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_6, \mathcal{B}_8\}$ be the set of students not at risk of dropping out. Then the lower and upper approximations and boundary regions of $\mathcal{K}$ are $L_{\mathcal{C}}(\mathcal{K}) = \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_6, \mathcal{B}_8\}$, $U_{\mathcal{C}}(\mathcal{K}) = \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_6, \mathcal{B}_8\}$ and $B_{\mathcal{C}}(\mathcal{K}) = \emptyset$. Therefore the nano topology is $\tau_{\mathcal{C}}(\mathcal{K}) = \{\emptyset, \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_6, \mathcal{B}_8\}, \mathcal{U}\}$ and its basis is $\beta_{\mathcal{C}}(\mathcal{K}) = \{\mathcal{U}, \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_6, \mathcal{B}_8\}, \emptyset\}$.

Step 1:

When the attribute "Academic Performance" is excluded from $\mathcal{C}$,

$\mathcal{U}/\mathcal{R}(\mathcal{C} - AP) = \{\{\mathcal{B}_1, \mathcal{B}_2\}, \{\mathcal{B}_3, \mathcal{B}_5, \mathcal{B}_6\}, \{\mathcal{B}_4\}, \{\mathcal{B}_7\}, \{\mathcal{B}_8\}\}$. Here $L_{\mathcal{C}-\{AP\}}(\mathcal{K}) = \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_8\}$, $U_{\mathcal{C}-\{AP\}}(\mathcal{K}) = \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_5, \mathcal{B}_6, \mathcal{B}_8\}$ and $B_{\mathcal{C}-\{AP\}}(\mathcal{K}) = \{\mathcal{B}_3, \mathcal{B}_5, \mathcal{B}_6\}$.

Therefore $\tau_{\mathcal{C}-\{AP\}}(\mathcal{K}) = \{\emptyset, \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_8\}, \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_5, \mathcal{B}_6, \mathcal{B}_8\}, \{\mathcal{B}_3, \mathcal{B}_5, \mathcal{B}_6\}, \mathcal{U}\}$ and its basis is $\beta_{\mathcal{C}-\{AP\}}(\mathcal{K}) = \{\mathcal{U}, \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_8\}, \{\mathcal{B}_3, \mathcal{B}_5, \mathcal{B}_6\}\}$.

Hence, $\beta_{\mathcal{C}-\{AP\}}(\mathcal{K}) \neq \beta_{\mathcal{C}}(\mathcal{K})$.

Step 2:

When the attribute "Student Engagement" is excluded from $\mathcal{C}$,

$\mathcal{U}/\mathcal{R}(\mathcal{C} - SE) = \{\{\mathcal{B}_1, \mathcal{B}_2\}, \{\mathcal{B}_3, \mathcal{B}_5, \mathcal{B}_7\}, \{\mathcal{B}_4\}, \{\mathcal{B}_6\}, \{\mathcal{B}_8\}\}$. Here

$L_{\mathcal{C}-\{SE\}}(\mathcal{K}) = \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_6, \mathcal{B}_8\}$, $U_{\mathcal{C}-\{SE\}}(\mathcal{K}) = \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_6, \mathcal{B}_8\}$ and $B_{\mathcal{C}-\{SE\}}(\mathcal{K}) = \emptyset$.

Therefore $\tau_{\mathcal{C}-\{SE\}}(\mathcal{K}) = \{\emptyset, \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_6, \mathcal{B}_8\}, \mathcal{U}\}$ and its basis is

$\beta_{\mathcal{C}-\{SE\}}(\mathcal{K}) = \{\mathcal{U}, \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_6, \mathcal{B}_8\}, \emptyset\}$.

Hence, $\beta_{\mathcal{C}-\{SE\}}(\mathcal{K}) = \beta_{\mathcal{C}}(\mathcal{K})$.

Step 4:

When the attribute "Health and Family" is excluded from $\mathcal{C}$,

$\mathcal{U}/\mathcal{R}(\mathcal{C} - HF) = \{\{\mathcal{B}_1, \mathcal{B}_2\}, \{\mathcal{B}_3, \mathcal{B}_5\}, \{\mathcal{B}_4\}, \{\mathcal{B}_6\}, \{\mathcal{B}_7\}, \{\mathcal{B}_8\}\}$.

Here $L_{\mathcal{C}-\{HF\}}(\mathcal{K}) = \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_6, \mathcal{B}_8\}$, $U_{\mathcal{C}-\{HF\}}(\mathcal{K}) = \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_6, \mathcal{B}_8\}$ and $B_{\mathcal{C}-\{HF\}}(\mathcal{K}) = \emptyset$.

Therefore $\tau_{\mathcal{C}-\{HF\}}(\mathcal{K}) = \{\emptyset, \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_6, \mathcal{B}_8\}, \mathcal{U}\}$ and its basis is

$\beta_{\mathcal{C}-\{HF\}}(\mathcal{K}) = \{\mathcal{U}, \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_6, \mathcal{B}_8\}, \emptyset\}$.

Hence, $\beta_{\mathcal{C}-\{HF\}}(\mathcal{K}) = \beta_{\mathcal{C}}(\mathcal{K})$.

Step 5:

When the attribute "Financial Impact" is excluded from $\mathcal{C}$,

$\mathcal{U}/\mathcal{R}(\mathcal{C} - FI) = \{\{\mathcal{B}_1, \mathcal{B}_2\}, \{\mathcal{B}_3, \mathcal{B}_5\}, \{\mathcal{B}_4\}, \{\mathcal{B}_6\}, \{\mathcal{B}_7, \mathcal{B}_8\}\}$.Here

$L_{\mathcal{C}-\{FI\}}(\mathcal{K}) = \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_6\}$, $U_{\mathcal{C}-\{FI\}}(\mathcal{K}) = \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_6, \mathcal{B}_7, \mathcal{B}_8\}$ and $B_{\mathcal{C}-\{FI\}}(\mathcal{K}) = \{\mathcal{B}_7, \mathcal{B}_8\}$.

Therefore $\tau_{\mathcal{C}-\{IF\}}(\mathcal{K}) = \{\emptyset, \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_6\}, \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_6, \mathcal{B}_7, \mathcal{B}_8\}, \{\mathcal{B}_7, \mathcal{B}_8\}, \mathcal{U}\}$ and its basis is $\beta_{\mathcal{C}-\{FI\}}(\mathcal{K}) = \{\mathcal{U}, \{\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_6\}, \{\mathcal{B}_7, \mathcal{B}_8\}\}$.

Hence, $\beta_{\mathcal{C}-\{FI\}}(\mathcal{K}) \neq \beta_{\mathcal{C}}(\mathcal{K})$.

Therefore CORE = {Academic Performance, Financial Impact}.

From the CORE of the above two cases, we observe that 'Academic Performance' and 'Financial Impact' are the key factors influencing student dropout risk.

Conclusion

This paper introduces a new class of closed sets, namely $\mathcal{N}_{\alpha}\mathcal{MC}$-closed sets, in nano topological spaces and investigates their fundamental properties. The relationships between $\mathcal{N}_{\alpha}\mathcal{MC}$-closed sets and other existing nano closed sets are established, and the associated interior and closure operators are studied. To illustrate the applicability of the proposed concept, an attribute reduction problem concerning student dropout risk in higher education is considered. Using nano topological bases, significant attributes are identified while redundant attributes are eliminated, thereby facilitating effective decision-making. The study demonstrates that $\mathcal{N}_{\alpha}\mathcal{MC}$-closed sets provide a useful framework for both theoretical investigations and practical applications. Future work may extend these concepts to other decision-making and knowledge discovery problems.

Acknowledgement

The authors are thankful to the anonymous referees for putting forth their valuable suggestions to refine this article.

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Conflict of Interest

The authors hereby declare no potential conflicts of interest with respect to the research, funding, authorship, and/or publication of this article