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Further characterizations and Helly-property in k-trees

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Abstract

The purpose of this paper is to obtain a characterization of k-trees in terms of k-connectivity and forbidden subgraphs. Also, we present the other characterizations of k-trees containing the full vertices by using the join operation. Further, we establish the property of k-trees dealing with the degrees and formulate the Helly-property for a family of nontrivial k-paths in a k-tree. We study the planarity of k-trees and express the maximal outerplanar graphs in terms of 2-trees and K_2-neighbourhoods. Finally, the similar type of results for the maximal planar graphs are obtained.

Keywords: Trees, Cycles, Paths, Connected graphs, Triangulated graphs, Planar graphs

Mathematics Subject Classification (2010): 05C10

1. Introduction

All graphs considered here are finite and simple.For any graph G, let $V(G)$ and $E(G)$ denote its vertex set and edge set, respectively. The order of G is $|V(G)|$ and its size is $|E(G)|$. A graph of order p and size q is a (p, q)-*graph*. For any two disjoint graphs G and H, $G + H$ denotes the *join* of G and H. All definitions and notations are not given here may be found in Harary[4]. A graph G is *n-connected* if the removal of any m vertices for $0 \leq m < n$, from G results in neither a disconnected graph nor a trivial graph. 1-connected graphs are simply the connected graphs. A graph G is *triangulated* if every cycle of length strictly greater than 3 possesses a chord. Any n mutually adjacent vertices i.e., K_n in a graph is *n-clique*. For any set S of vertices of a graph G, $\langle S \rangle$ denotes the *induced subgraph* of G induced by S. For

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any connected graph G, nG denotes the graph with n components, each being isomorphic to G.

A family of trees, which are connected and acyclic, can be equivalently defined by the following recursive construction rule:
Step 1. A single vertex K_1 is a tree.
Step 2. Any tree of order $n \geq 2$, can be constructed from a tree T of order $(n-1)$ by inserting an n^{th}-vertex and joining it to any vertex of T.

More general, the multidimensional-trees can be constructed from the above tree-construction procedure by allowing the base of the recursive growth to be any clique. Notice that a connected graph, which is not a tree possesses a tree-like structure, which is actually reflected by constructing the new family of graphs, whose recursive growth just starts from any given clique K_k. This family of graphs are generally known as k-trees or K_k-trees or k-dimensional trees.[1, 5, 7, 8]

Definition 1.1. *The family of* k-trees (or K_k-trees) *is the set of all graphs that can be obtained by the following recursive construction procedure :*
***1**. A clique-K_k is the smallest k-tree.*
***2**. To a k-tree G with $n-1$ vertices for $n \geq k+1$, add a new vertex and make it adjacent to any k mutually adjacent vertices of G, so that the resulting k-tree is of order n.*

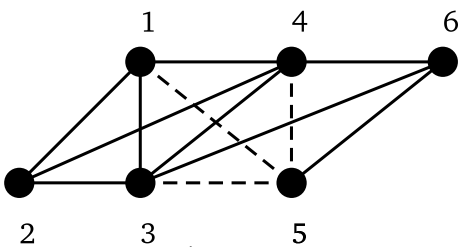

Figure 1

Figure 1 gives the example of a 3-tree of order 6. Generally speaking, every k-tree G of order $\geq k+1$, can be reduced to a clique K_k, by sequentially removing the vertices of degree k from G.

2. Properties and Characterizations

We need the following characterization theorem for later use.

Theorem 2.1. *[5] Let G be a (p, q)-graph with $p \geq k+1$. Then G is a k-tree if and only if G is k-connected, triangulated and either G is K_{k+2}-free or $q = (kp - \frac{k(k+1)}{2})$.*

The immediate consequence of **Theorem 2.1** is another characterization of k-trees in terms of forbidden subgraphs and k-connectivity.

Corollary 2.2. *Let G be a graph of order at least $k+1$. Then G is a k-tree if and only if G is k-connected and has no induced subgraph isomorphic to either C_n for $n \geq 4$ or K_{k+2}.*

We first obtain the basic property of k-trees dealing with degrees. For this, we need to establish the following lemma.

Lemma 2.3. *Every k-connected, (p, q)-graph G with $p \geq k+1$ and $q = (kp - \frac{k(k+1)}{2})$, has at least $k+1$ vertices, whose degrees do not exceed $2k-1$.*

Proof. Since G is k-connected, $deg\ v_i \geq k$ for all v_i in $V(G)$. Let t be the number of vertices in G, whose degrees are at most $2k-1$. Consequently, G contains $p-t$ vertices of degrees at least $2k$. Immediately, we have

$$\sum_{i=1}^{p} deg\ v_i \geq tk + (p-t)\,2k. \tag{1}$$

On the other hand, by the *handshaking theorem,* we have

$$\sum_{i=1}^{p} deg\ v_i = 2q = 2(kp - \frac{k(k+1)}{2}). \tag{2}$$

From equations (1) and (2), we have

$$2kp - k(k+1) \geq tk + (p-t)2k.$$

This shows that $t \geq k+1$ and hence, G contains at least $k+1$ vertices, whose degrees do not exceed $2k-1$. □

The direct consequence of **Lemma 2.3** is the following result. Moreover, for $k = 1$, this result extends the property of trees (**Corollary 4.1 (a)** p.34, [4]).

Corollary 2.4. *Every k-tree of order at least $k+1$, has at least $k+1$ vertices, whose degrees do not exceed $2k-1$.*

Proof. Let G be a k-tree of order $p \geq k+1$. By Theorem 2.1, G is a triangulated, k-connected graph of size $(kp - \frac{k(k+1)}{2})$. From **Lemma 2.3**, the result follows. □

Next, we show that the bound given in **Corollary 2.4**, is the best possible by constructing below a k-tree G with exactly $k+1$ vertices, whose degrees do not exceed $2k-1$. Let G be a graph consists of $K_{k+1} \cup \overline{K_{k+1}}$, with all the possible additional edges $u_i v_j$ for $i \neq j$, where u_i and v_j are the vertices in K_{k+1} and $\overline{K_{k+1}}$, respectively (for $1 \leq i,\ j \leq k+1$). Now, we observe that G is a k-tree of order $2k+2$ and it contains $k+1$ vertices of degree k and $k+1$ vertices of degree $2k$.

Definition 2.5. *Let G be a graph of order p. A vertex v in G is called a* full-vertex *if $deg\ v = p - 1$.*

For example, $K_k + \overline{K_{p-k}}$ (for $k < p$), is a k-tree of order p, containing exactly k full-vertices. We now obtain a characterization of k-trees containing at least one full-vertex.

Theorem 2.6. *Let G be a graph of order $p \geq k + 1$. Then G is a k-tree containing a full-vertex if and only if G is isomorphic to $K_1 + H$, where H is a $(k - 1)$-tree of order $p - 1$.*

Proof. Suppose that G is a k-tree, containing a full-vertex v. By Theorem 2.1, G is a k-connected, triangulated graph of size $(kp - \frac{k(k+1)}{2})$. Let $\langle\{v\}\rangle \cong K_1$. Since $deg\ v = p - 1$ in G, the removal of v from G certainly reduces its connectivity by one, without affecting its triangularity property and further, we have

$$|E(G - v)| = (kp - \frac{k(k+1)}{2}) - (p - 1) = (k - 1)(p - 1) - \frac{k(k-1)}{2}.$$

From Theorem 2.1, $G - v$ is a $(k - 1)$-tree of order $p - 1$. However, we see that G is isomorphic to $K_1 + (G - v)$.

Conversely, assume that G is isomorphic to $K_1 + H$, where H is a $(k - 1)$-tree of order $p - 1$. Since $deg\ v = p - 1$ in G, it follows that H is isomorphic to $G - v$. Consequently, $G = K_1 + (G - v)$ is a k-connected, triangulated graph of size $(kp - \frac{k(k+1)}{2})$. By Theorem 2.1, G is a k-tree. □

Repeated application of **Theorem 2.6**, yields the general criterion for k-trees containing at most k full-vertices.

Corollary 2.7. *Let G be a graph of order $p \geq k + 1$. Then G is a k-tree containing t full-vertices $(1 \leq t \leq k)$ if and only if G is isomorphic to $K_t + T_{p-t}$, where T_{p-t} is a $(k - t)$-tree of order $p - t$ and T_{p-k} is a forest.*

3. Helly-property on k-paths

We begin with the notion of m-walk for $m \geq 2$, which extends the concept of a walk (i.e., 1-walk) introduced by Beineke and Pippert.[1]

Definition 3.1. ***(1).*** *A m-*walk *for $m \geq 1$, in a graph G, denoted by $W(K_m^0, K_m^n)$; $n \geq 0$, is an alternating finite sequence of its distinct cliques K_m and K_{m+1} of the form:*
$(K_m^0, K_{m+1}^1, K_m^1, K_{m+1}^2, \ldots, K_m^{n-1}, K_{m+1}^n, K_m^n)$, beginning and ending with the cliques K_m^0 and K_m^n, respectively such that for each i $(1 \leq i \leq n)$, $K_{m+1}^i = K_m^{i-1} \cup K_m^i$ and $K_m^{i-1} \cap K_m^i = K_{m-1}$.
(2). *A m-walk $W(K_m^0, K_m^n)$; $n \geq 0$, is called a m-*path *if all its cliques*

$K_m^0, K_m^1, \ldots, K_m^n$ *and* $K_{m+1}^1, K_{m+1}^2, \ldots, K_{m+1}^n$ *are distinct. The* length *of a m-path, is the number of occurrences of cliques* K_{m+1} *in it. For example, any clique* K_m *is a* trivial *m-path ;* K_{m+1} *is a* nontrivial *m-path of length 1;* $K_m + \overline{K_2}$ *is a* nontrivial *m-path of length 2.*

In Figure 2, the anatomy of a 2-path is shown.

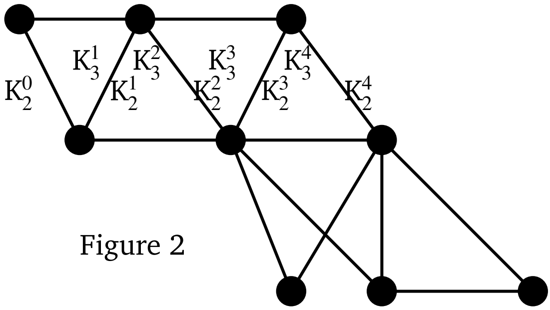

Figure 2

Let $\Pi = \{J_i \ : \ i \in I\}$ be a family of subsets of a finite set S (where I denotes the index set). Then Π is said to satisfy the *Helly-property* if $J_i \cap J_j \neq \emptyset$ for all i, j in I, implies that $\cap_{k \in I} J_k \neq \emptyset$.

For example, $\Pi = \{J_1, J_2, J_3\}$, where the nontrivial paths : $J_1 = abc$; $J_2 = cbd$; $J_3 = abd$, of the tree $K_{1,3}$ as shown in Figure 3.

Notice that every two paths in Π have a nontrivial intersection, but there is no common nontrivial path for all three paths in Π.

We now establish the Helly-property for a family of nontrivial k-paths of a k-tree.

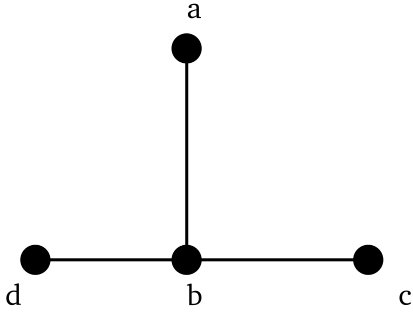

Figure 3

Proposition 3.2. *Let* $\Pi = \{J_i \ : \ i \in I\}$ *be a finite family of nontrivial k-paths of a k-tree. If every three k-paths* J_i, J_j, J_k *for* $i, j, k \in I$*, have a nontrivial intersection, then* $\cap_{n \in I} J_n$ *is a nontrivial intersection.*

Proof. Let G be a k-tree. We prove the result by induction on the number of nontrivial k-paths of G. Assume that $\cap_{n \in J} J_n$ is isomorphic to W,

where $|J| = t < |I|$; J is an index set, is a nontrivial k-path of G.
If J_{t+1} has no nontrivial intersection with W, then there exist always three k-paths J_{t+1}, J_t and J_{t-1} of G, which have no nontrivial intersection. (In fact, for $k = 1$, this fact is illustrated in Figure 4). This is a contradiction to the hypothesis. Hence, the desired property is proved. □

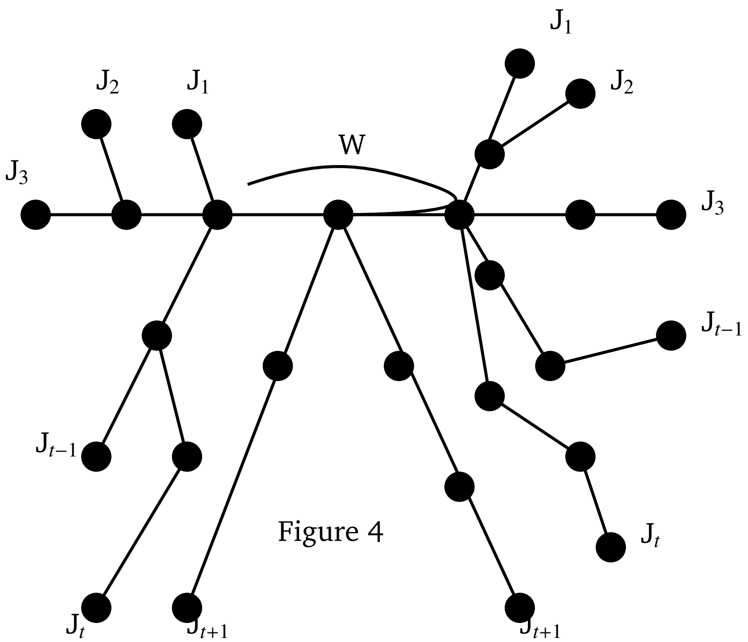

Figure 4

4. Planarity and Clique-neighbourhoods

The *neighbourhood* of a vertex u in a graph G is the set $N(u)$ consisting of all the vertices, which are adjacent to u. A vertex u is *simplicial* if $N(u)$ induces a clique in G.

Definition 4.1. *For any clique K_p of a graph G with vertices $u_1, u_2, u_3, \ldots, u_p$, the K_p*-neighbourhood, *denoted by $N(K_p)$ is $\cap_{i=1}^{p} N(u_i)$.*

Notice that 1-trees (i.e., trees) are obviously planar. The maximal outerplanar graphs are the special class of 2-trees. The triangulated, maximal planar graphs are restricted family of 3-trees. All nontrivial 4-trees (other than K_4) and k-trees ($k \geq 5$) are nonplanar. To study (outer)planarity, let us first establish the following lemma.

Lemma 4.2. *Let G be a k-tree of order $\geq k+1$. For any clique K_k in G,*
a). *$N(K_k) \neq \emptyset$.*
b). *$N(K_k)$ is an independent set.*

Proof. To prove (a), we use the induction on order $p \geq k+1$ of G. If $p = k+1$, then $G = K_{k+1}$. Obviously, $|N(K_k)| = 1$ for any clique K_k in G and hence the result is obvious. We assume that the result holds for any $p : k+2 \leq p \leq n$. Let G be a k-tree with $p = n+1$. Then by Definition 1.1, G contains a simplicial vertex u of degree k and $G - u$ is a k-tree of order n. By induction hypothesis, $N(K_k) \neq \emptyset$ for any clique K_k in $G - u$. Let $N(u) = \{u_1, u_2, \ldots, u_k\}$ and $N(u)$ is isomorphic to K_k. Consider any clique K_k^i of G with $V(K_k^i) = \{u\} \cup (N(u) - \{u_i\})$ for $1 \leq i \leq k$. Immediately, we observe that $N(K_k^i) = \{u_i\}$. Thus, $N(K_k^i) \neq \emptyset$. By induction, the result follows for all $p \geq k+1$.
To prove (b), if possible, we assume that for some clique K_k in G, $N(K_k)$ is not independent. Then G contains at least two vertices u and v in $N(K_k)$ such that u and v are adjacent in G. This shows that $\langle N(u) \cup \{u, v\}\rangle$ is isomorphic to K_{k+2} in G. This is not possible (by Theorem 2.1), because G is a k-tree. □

In [5], it is proved that any graph G of order ≥ 3, is maximal outerplanar if and only if G is 2-connected, triangulated and outerplanar. Next, we present another characterization of a maximal outerplanar graph involving 2-trees and K_2-neighbourhoods.

Proposition 4.3. *Let G be a graph of order ≥ 3. Then G is maximal outerplanar if and only if G is a 2-tree and for any complete graph K_2 of G, $\langle N(K_2)\rangle$ is either K_1 or $2K_1$.*

Proof. Suppose that G is maximal outerplanar. Immediately, G is 2-connected, triangulated and outerplanar. Since G is outerplanar, G is K_4-free. By Theorem 2.1 with $k = 2$, G is a 2-tree. On contrary, assume that $|N(K_2)| \geq 3$ for some complete graph K_2 of G. Let x, y and z be the vertices in $N(K_2)$. Consequently, $\langle\{u, v, x, y, z\}\rangle$ isomorphic to $K_2 + 3K_1$ appears in G. But $K_2 + 3K_1$ contains a subgraph isomorphic to $K_{2,3}$ and hence G is not outerplanar. This leads to a contradiction. So, $|N(K_2)| \leq 2$ for each complete graph K_2 of G. From Lemma 4.1 with $k = 2$, we have $|N(K_2)| \geq 1$ and $\langle N(K_2)\rangle$ is either K_1 or $2K_1$. Necessity is thus proved.

It is easy to prove the converse. □

The immediate consequence of the above proposition is Corollary 11.9 (a) of [4, p. 107]. Certainly, this bound can be improved for nonouterplanar, 2-trees.

Corollary 4.4. *Every 2-tree other than maximal outerplanar, has at least three vertices of degree 2.*

Proof. Follows from the immediate consequence of Proposition 4.3. □

Notice that a maximal planar graph need not be triangulated. For example, $C_4 + 2K_1$ is maximal planar but not triangulated.

Proposition 4.5. *Let G be a triangulated graph of order ≥ 4. Then G is maximal planar if and only if G is a 3-tree and for any triangle K_3 in G, $\langle N(K_3) \rangle$ is either K_1 or $2K_1$.*

The proof follows on the similar arguments as used in the proof of Proposition 4.3, by using Theorem 2.1 with $k = 3$.

The following corollary is the immediate consequence of the above result.

Corollary 4.6. *Every nonplanar 3-tree, has at least three vertices of degree 3.*

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