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Vertex Triangle Free Detour Number of a Graph

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Abstract

The *x-triangle free detour number* $dn_{\triangle f_x}(G)$ of a connected graph G is the minimum order of its x-triangle free detour sets and any x-triangle free detour set $S_x \subseteq V$ of order $dn_{\triangle f_x}(G)$ is a *x-triangle free detour basis* of G. A connected graph of order n with vertex triangle free detour number $n-1$ or $n-2$ for every vertex is characterized. Certain general properties satisfied by the vertex triangle free detour sets are studied.

Keywords: Triangle free detour distance, Triangle free detour number, Vertex triangle free detour set, Vertex triangle free detour number

Mathematics Subject Classification (2010): 05C12

1. Introduction

The concept of *triangle free detour distance* was introduced by Keerthi Asir and Athisayanathan.[3] A path P is called a *triangle free path* if no three vertices of P induce a triangle. For vertices u and v in a connected graph G, the *triangle free detour distance* $D_{\triangle f}(u, v)$ is the length of a longest $u-v$ triangle free path in G. A $u-v$ path of length $D_{\triangle f}(u, v)$ is called a *$u-v$ triangle free detour*. The triangle free detour eccentricity $e_{\triangle f}(v)$ of a vertex in G is the maximum triangle free detour distance from v to a vertex of G. The triangle free detour radius, $rad_{\triangle f}(G)$ or $R_{\triangle f}$ of G is the minimum triangle free detour eccentricity among the vertices of G, while the triangle free detour diameter, $diam_{\triangle f}(G)$ or $D_{\triangle f}$ of G is the maximum triangle free detour eccentricity among the vertices of G.

The concept of *triangle free detour number* was introduced and studied by Sethu Ramalingam *et al.*[6] A set $S \subseteq V$ is called a *triangle free*

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detour set of G if every vertex of G lies on a triangle free detour joining a pair of vertices of S. The *triangle free detour number* $dn_{\triangle f}(G)$ of G is the minimum order of its triangle free detour sets and any triangle free detour set of order $dn_{\triangle f}(G)$ is called a *triangle free detour basis* of G.
The concept of *vertex detour number* of a graph was introduced and studied in [4]. For any vertex x in a connected graph G, a set S of vertices of G is an *x-detour set* if each vertex v of G lies on an $x-y$ detour in G for some vertex y in S. The minimum cardinality of an x-detour set of G is defined as the *x-detour number* of G, denoted by $d_x(G)$ or simply d_x. An x-detour set of cardinality $d_x(G)$ is called a d_x-set of G. The concept of *vertex detour monophonic number* of a graph was introduced and studied by Titus and Balakrishnan.[7] A *chord* of a path P is an edge joining two non-adjacent vertices of P. A path P is called *monophonic* if it is a chordless path. A longest $u-v$ monophonic path is called an *$u-v$ detour monophonic path*. For any vertex x in a connected graph G, a set S of vertices of G is an *x-detour monophonic set* if each vertex v of G lies on an $x-y$ detour monophonic in G for some vertex y in S. The minimum cardinality of an x-detour monophonic set of G is defined as the *x-detour monophonic number* of G, denoted by $dm_x(G)$ or simply dm_x. An x-detour monophonic set of cardinality $dm_x(G)$ is called a dm_x-set of G.

The concept of *vertex geodetic number* of a graph was introduced and studied by Santhakumaran *et al.*[5] For any vertex x in a connected graph G, a set S of vertices of G is an *x-geodetic set* if each vertex v of G lies on an $x-y$ geodetic in G for some vertex y in S. The minimum cardinality of an x-geodetic set of G is defined as the *x-geodetic number* of G, denoted by $g_x(G)$ or simply g_x. An x-geodetic set of cardinality $g_x(G)$ is called a g_x-set of G. Throughout this paper G denotes a finite undirected simple connected graph with at least two vertices. For basic definitions and terminologies, we refer to Chartrand and Zhang.[1]

The following theorems are useful for the results in this paper.

Theorem 1.1. *[2] Let v be a vertex of a connected graph G. The following statements are equivalent:*
(i) v is a cut vertex of G.
(ii) There exist vertices u and w distinct from v such that v is on every $u-w$ path.
(iii) There exists a partition of the set of vertices $V-\{v\}$ into subsets U and W such that for any vertices $u \in U$ and $w \in W$, the vertex v is on every $u-w$ path.

Theorem 1.2. *[6] Every extreme-vertex of a connected graph G belongs to every triangle free detour set of G.*

Theorem 1.3. *[6] If T is a tree with k end-vertices, then $dn_{\triangle f}(T) = k$.*

Theorem 1.4. *[4] Let x be any vertex of a connected graph G.*
(i) Every end-vertex of G other than the vertex x(whether x is end-vertex or not) belong to every x-detour set.
(ii) No cut vertex of G belongs to any d_x-set.

Theorem 1.5. *[7] Let x be any vertex of a connected graph G.*
(i) Every end-vertex of G other than the vertex x(whether x is end-vertex or not) belong to every x-detour monophonic set.
(ii) No cut vertex of G belongs to any dm_x-set.

Theorem 1.6. *[5] Let x be any vertex of a connected graph G.*
(i) Every end-vertex of G other than the vertex x (whether x is end-vertex or not) belong to every x-geodetic set.
(ii) No cut vertex of G belongs to any g_x-set.

2. Vertex Triangle Free Detour Number

Let x be a vertex of a connected graph G. A set $S_x \subseteq V$ is called an x-triangle free detour set of G if every vertex v of G lies on a $x-y$ triangle free detour in G for some vertex y in S_x. The vertex triangle free detour number $dn_{\triangle f_x}(G)$ of G is the minimum order of its x-triangle free detour sets and any x-triangle free detour set of order $dn_{\triangle f_x}(G)$ is a vertex triangle free detour basis of G. An x-triangle free detour set of cardinality $dn_{\triangle f_x}(G)$ is called a $dn_{\triangle f_x}$-set of G.

Theorem 2.1. *For any vertex x in G, x does not belong to any $dn_{\triangle f_x}$-set of G.*

Proof. Suppose that x belongs to a $dn_{\triangle f_x}$-set, say S_x of G. Since G is a connected graph with at least two vertices, it follows from the definition of an x-triangle free detour set that S_x contains a vertex v different from x. Since the vertex x lies on every $x-v$ triangle free detour in G, it follows that $T = S_x - \{x\}$ is an x-triangle free detour set of G, which is a contradiction to S_x a minimum x-triangle free detour set of G. □

Example 2.2. *For the graph G given in Figure 2.1, a minimum vertex triangle free detour sets and the vertex triangle free detour numbers are given in Table 2.1.*

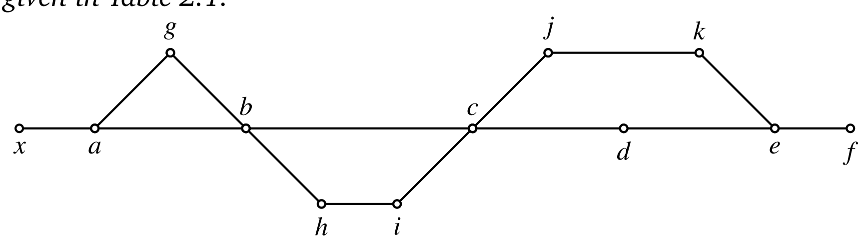

Figure 2.1: G

For the graph G given in Figure 2.1, the sets $S_1 = \{d, f\}$, $S_2 = S_1 \cup \{g\}$, $S_3 = S_2 \cup \{h\}$ and $S_4 = S_3 \cup \{j\}$ are minimum x-detour set, minimum x-triangle free detour set, minimum x-detour monophonic set and minimum x-geodetic set respectively and hence $d_x(G) = 2$, $dn_{\triangle f_x}(G) = 3$, $dm_x(G) = 4$ and $g_x(G) = 5$. Thus the vertex detour number, vertex triangle free detour number, vertex detour monophonic number and vertex geodetic number of a graph G are distinct.

Vertex t	Minimum $dn_{\triangle f_t}$-set	$dn_{\triangle f_t}(G)$
x	$\{g, d, f\}$	3
a	$\{x, g, d, f\}$	4
b	$\{x, g, d, f\}$	4
c	$\{x, g, d, f\}$	4
d	$\{x, g, f\}$	3
e	$\{x, g, d, f\}$	4
f	$\{x, g, d\}$	3
g	$\{x, d, f\}$	3
h	$\{x, g, d, f\}$	4
i	$\{x, g, d, f\}$	4
j	$\{x, g, d, f\}$	4
k	$\{x, g, d, f\}$	4

Table 2.1

Remark 2.3. *Let x be any vertex of G. Then for any vertex y belongs to a $dn_{\triangle f_x}$-set S_x of G, the internal vertices of an $x - y$ triangle free detour may belong to S_x. For the graph G given in Figure 2.2, $S_x = \{w, z\}$ is a $dn_{\triangle f_x}$-set of G and u belongs to S_x is an internal vertex of the $x - w$ triangle free detour say P: x, u, v, w.*

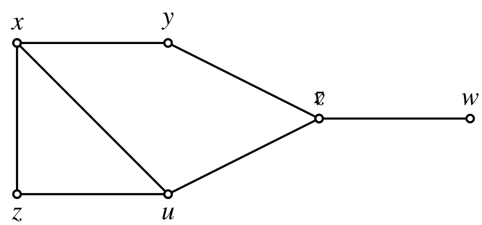

Figure 2.2: G

Theorem 2.4. *Let x be any vertex of a connected graph G.*
(i) *Every end-vertex of G other than the vertex x(whether x is end-vertex or not) belong to every x-triangle free detour set.*
(ii) *No cut vertex of G belongs to any $dn_{\triangle f_x}$-set.*

Proof. (i) Let x be any vertex of G. By Theorem 2.1, x does not belong to any $dn_{\triangle f_x}$-set. So let $v \neq x$ be an end-vertex of G. Then v is the terminal vertex of an $x-v$ triangle free detour and v is not an internal vertex of any triangle free detour so that v belongs to every x-triangle free detour set of G.

(ii) Let y be a cut vertex of G. Then by Theorem 1.1, there exists a partition of the set of vertices $V - \{y\}$ into subsets U and W such that for any vertex $u \in U$ and $w \in W$, the vertex y is on every $u - w$ path. Hence, if $x \in U$, then for any vertex w in W, y lies on every $x - w$ path so that y is an internal vertex of an $x - w$ triangle free detour. Let S_x be any $dn_{\triangle f_x}$ - set of G. Suppose $S_x \cap W = \phi$. Let $w_1 \in W$. Since S_x is an x-triangle free detour set, there exists an element z in S_x such that w_1 lies in some $x-z$ triangle free detour $P : x = z_0, z_1, ..., w_1, ..., z_n = z$ in G. Then the $x-w_1$ subpath of P and w_1-z subpath of P both contain y so that P is not a path in G. Hence $S_x \cap W \neq \phi$. Let $w_2 \in S_x \cap W$. Then y is an internal vertex of an $x - w_2$ triangle free detour. If $y \in S_x$, let $S = S_x - \{y\}$. It is clear that every vertex that lies on $x - y$ triangle free detour also lies on an $x-w_2$ triangle free detour. Hence it follows that S is an x-triangle free detour set of G, which is a contradiction to S_x is a minimum x-triangle free detour set of G. Thus y does not belong to any $dn_{\triangle f_x}$-set. Similarly if $x \in W$, y does not belong to any $dn_{\triangle f_x}$-set. If $x = y$, then by Theorem 2.1, y does not belong to any $dn_{\triangle f_x}$-set. □

Remark 2.5. *If x is an end-vertex of G, x does not belong to any $dn_{\triangle f_x}$-set by Theorem 2.1.*

Corollary 2.6. *Let T be a tree with t end-vertices. Then $dn_{\triangle f_x}(T) = t-1$ or $dn_{\triangle f_x}(T) = t$ according to whether x is an end-vertex or not. In fact, if W is the set of all end-vertices of T, then $W - \{x\}$ is the unique $dn_{\triangle f_x}$-set of T.*

Proof. Let W be the set of all end-vertices of T. It follows from Theorem 2.1 and Theorem 2.4 that $W - \{x\}$ is the unique $dn_{\triangle f_x}$-set of T for any end-vertex x in T and W is the unique $dn_{\triangle f_x}$-set of T for any cut vertex x in T. Thus $W - \{x\}$ is the unique $dn_{\triangle f_x}$-set of T for any vertex x in T. □

Theorem 2.7. *For any hamiltonian graph G, $dn_{\triangle f_x}(G) = 1$ for every vertex x in G.*

Proof. Let C be a hamiltonian cycle of G. Let x be any vertex of G and let y be any adjacent vertex of x in G. Clearly every vertex of G lies on a triangle free detour joining x and y. Thus $dn_{\triangle f_x}(G) = 1$ for every vertex x in G. □

Remark 2.8. *The converse of Theorem 2.7 is false. For the graph G given in Figure 2.3, $dn_{\triangle f_x}(G) = 1$ for every vertex x in G. But G is not hamiltonian.*

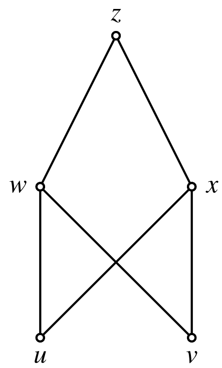

Figure 2.3: G

The following theorem is an easy consequence of the definition of the vertex triangle free detour number.

Theorem 2.9. (i) *For any path P_n, $dn_{\triangle f_x}(P_n) = 1$ or $dn_{\triangle f_x}(P_n) = 2$ according as x is an end-vertex or not.*
(ii) *For any cycle C_n, $dn_{\triangle f_x}(C_n) = 1$ for every vertex x in C_n.*
(iii) *For the wheel $W_n = K_1 + C_{n-1} (n \geq 5)$, $dn_{\triangle f_x}(W_n) = n-1$ or $dn_{\triangle f_x}(W_n) = 2$ according as x is in K_1 or x in C_{n-1}.*
(iv) *For every vertex x in G, $dn_{\triangle f_x}(K_{1,m}) = m$ or $dn_{\triangle f_x}(K_{n,m}) = m - 1$ if $m \geq 2$.*
(v) *For any complete graph K_n, $dn_{\triangle f_x}(K_n) = n-1$ for every vertex x in K_n.*
(vi) *For every vertex x in G, $dn_{\triangle f_x}(K_{n,m}) = 1$ if $n = m = 1$.*

Theorem 2.10. *Let G be a connected graph with cut vertices and let S_x be an x-triangle free detour set of G. Then every branch of G contains an element of $S_x \cup \{x\}$.*

Proof. Suppose that there is a branch B of G at a cut vertex v such that B contains no vertex of $S_x \cup \{x\}$. Then clearly, $x \in V - (S_x \cup V(B))$. Let $u \in V(B) - \{v\}$. Since S_x is an x-triangle free detour set, there is an element $y \in S_x$ such that u lies in some $x - y$ triangle free detour $P : x = u_0, u_1, ..., u, ..., u_n = y$ in G. By Theorem 1.1 the $x - u$ subpath of P and $u - y$ subpath of P both contain v, and it follows that P is not a path, contrary to assumption. □

Since every end-block B is a branch of G at some cut-vertex, it follows by Theorems 2.4 and 2.10 that every $dn_{\triangle f_x}$-set of G together with the vertex x contains at least one vertex from B that is not a cut-vertex. Thus the following corollaries are consequences of Theorem 2.10.

Corollary 2.11. *If G is a connected graph with k end-blocks, then $dn_{\triangle f_x}(G) \geq k-1$ for every vertex x in G.*

Theorem 2.12. *For any vertex x in G,* $1 \leq dn_{\triangle f_x}(G) \leq n-1$.

Proof. It is clear from the definition of $dn_{\triangle f_x}$-set that $dn_{\triangle f_x}(G) \geq 1$. Also since the vertex x does not belong to any $dn_{\triangle f_x}$-set, it follows that $dn_{\triangle f_x}(G) \leq n-1$. □

Remark 2.13. *The bounds in Theorem 2.13 are sharp. For the cycle C_n, $dn_{\triangle f_x}(C_n) = 1$ for every vertex x in C_n. Also for any path P_n, $dn_{\triangle f_x}(P_n) = 1$ for any end-vertex x in P_n. For the graph K_n, $dn_{\triangle f_x}(K_n) = n-1$ for every vertex x in K_n.*

In the following theorem, we establish the relationship between the vertex triangle free detour number of a graph of a vertex and the triangle free detour number of a graph.

Theorem 2.14. *For any vertex x in G,* $2 \leq dn_{\triangle f}(G) \leq dn_{\triangle f_x}(G) + 1$.

Proof. A triangle free detour set needs at least two vertices so that $dn_{\triangle f}(G) \geq 2$. Let x be any vertex of G and let S_x be a $dn_{\triangle f_x}$-set of G. Then every vertex of G lies on an $x-y$ triangle free detour for some y in S_x. Thus $S_x \cup \{x\}$ is a triangle free detour set of G. Since $dn_{\triangle f}(G)$ is the minimum cardinality of a triangle free detour set, it follows that $dn_{\triangle f}(G) \leq \mathrm{dn}_{\triangle f_x}(G) + 1$. □

Remark 2.15. *The bound in Theorem 2.14 is sharp. For the complete graph K_n, $dn_{\triangle f}(K_n) = dn_{\triangle f_x}(K_n) + 1$ for every vertex x in K_n.*

Theorem 2.16. *For any two integers a and b with $2 \leq a \leq b+1$, there exists a connected graph G with $dn_{\triangle f}(G) = a$ and $dn_{\triangle f_x}(G) = b$ for some vertex x in G.*

Proof. **Case 1.** $2 \leq a = b+1$. Let G be any tree with a end-vertices. By Theorem 1.3, $dn_{\triangle f}(G) = a$ and by Corollary 2.6, $dn_{\triangle f_x}(G) = b$ for an end-vertex x in G.
Case 2. $2 \leq a < b+1$. Let $F = (P_3 \cup P_2 \cup (b-a+1)K_1) + \overline{K_2}$, where $U = V(P_3) = \{u_1, u_2, u_3\}$, $W = V(P_2) = \{w_1, w_2\}$, $X = V((b-a+1)K_1) = \{x_1, x_2, ..., x_{b-a+1}\}$ and $V(\overline{K_2}) = \{x, y\}$. Let G be the graph obtained from F by adding $a-1$ new vertices $z_1, z_2, ..., z_{a-1}$ and joining each $z_i (1 \leq i \leq a-1)$ to u_1. The graph G is shown in Figure 2.4. Let $Z = \{z_1, z_2, ..., z_{a-1}\}$ be the set of end vertices of G.

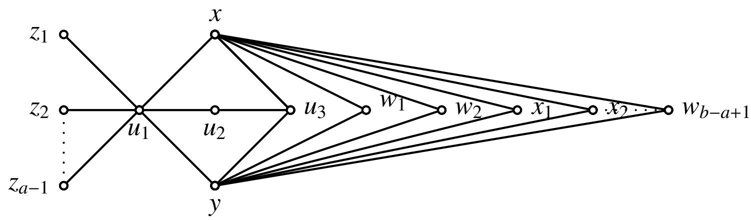

Figure 2.4: G

First, we show that $dn_{\triangle f}(G) = a$. By Theorem 1.2, every triangle free detour set of G contains Z. Since $Z \cup \{u_1\} \neq V(G)$, it follows that Z is not a triangle free detour set of G and so that $dn_{\triangle f}(G) > |Z| = a - 1$. On the other hand, let $S = Z \cup \{w_1\}$. Then $D_{\triangle f}(z_1, w_1) = 7$ and for each i with $1 \leq i \leq b - a + 1$, the path $z_1, u_1, u_2, u_3, y, x_i, x, w_1$ is a $z_1 - w_1$ triangle free detour in G. Hence S is a triangle free detour set of G and so $dn_{\triangle f}(G) \leq |S| = a$. Therefore $dn_{\triangle f}(G) = a$.

Next we show that $dn_{\triangle f_x}(G) = b$ for the vertex x. Let S_x be a minimum x-triangle free detour set of G. By Theorem 2.5 (i), $Z \subseteq S_x$. Since $D_{\triangle f}(x, Z) = 6$ and no $x_i (1 \leq i \leq b - a + 1)$ lies on an $x - z$ triangle free detour for any $z \in Z$, Z is not an x-triangle free detour set of G. Now we claim that $X \subseteq S_x$. Assume, to the contrary, $X \nsubseteq S_x$. Then there exists an x_i such that $x_i \notin S_x (1 \leq i \leq b - a + 1)$. Now this x_i does not lie on any $x - v$ triangle free detour for $v \neq x_i$ and $v \in S_x$, this is a contradiction to S_x is a x-triangle free detour basis. Thus $X \subseteq S_x$. It is clear that $X \cup Z$ is an x-triangle free detour set. Hence it follows that $X \cup Z$ is an x-triangle free detour basis so that $dn_{\triangle f_x}(G) = a - 1 + b - a + 1 = b$. □

3. Bounds for the Vertex Triangle Free Detour Number of a Graph

Theorem 3.1. *For any vertex x in a connected graph G of order n and a triangle free detour eccentricity $e_{\triangle f}(x)$, $dn_{\triangle f_x}(G) \leq n - e_{\triangle f}(x)$.*

Proof. Let x be any vertex of G and v a triangle free detour eccentric vertex of x. Then $D_{\triangle f}(u, v) = e_{\triangle f}(x)$. Let $P : x = x_0, x_1, ..., x_k = v$ be an $x - v$ triangle free detour in G. Let $S = V(G) - \{x_0, x_1, ..., x_{k-1}\}$. Since each x_i $(0 \leq i \leq k - 1)$ lies on an $x - v$ triangle free detour, S is an x-triangle free detour set of G so that $dn_{\triangle f_x}(G) \leq n - e_{\triangle f}(x)$. □

Remark 3.2. *The bounds in Theorem 3.1 is sharp. For the cycle C_n, $dn_{\triangle f_x}(C_n) = 1 = n - e_{\triangle f}(x)$ for every vertex x in C_n. Also for the graph G in Figure 3.1, $n = 10$, $e_{\triangle f}(x_7) = 7$ and $S = \{x_4, x_9, x_{10}\}$ is a $dn_{\triangle f_{x_7}}$-set so that $dn_{\triangle f x_7} = 3$. Thus $dn_{\triangle f x_7} = n$ - $e_{\triangle f}(x_7)$. The inequality in Theorem 3.1 can also be strict. For the same graph G given in Figure 3.1, $e_{\triangle f}(x_3) = 5$ and $S = \{x_4, x_7, x_9, x_{10}\}$ is a $dn_{\triangle f x_3}$-set so that $dn_{\triangle f x_3}(G) = 4$. Thus $dn_{\triangle f x_3}(G) < n - e_{\triangle f}(x_3)$.*

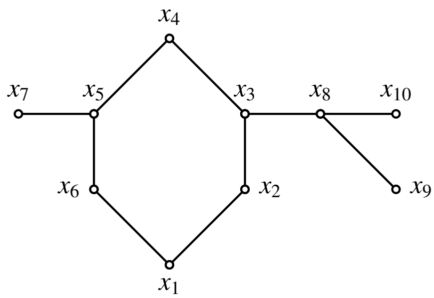

Figure 3.1: G

Corollary 3.3. *If G is a connected graph of order n and triangle free detour diameter $D_{\triangle f}$, then $dn_{\triangle f_x}(G) \le n - \frac{D_{\triangle f}}{2}$ for every vertex x in G.*

Proof. Since $R_{\triangle f} \le e_{\triangle f}(x)$ for every vertex x in G and Theorem 3.1 that $dn_{\triangle f_x}(G) \le n - \frac{D_{\triangle f}}{2}$. □

Remark 3.4. *The bound in Corollary 3.3 is sharp. For the star $K_{1,n-1}(n \ge 3)$, by Theorem 2.10(iv), $dn_{\triangle f_x}(K_{1,n}) = n - 1 = n - \frac{D_{\triangle f}}{2}$ for the cut vertex x in $K_{1,n-1}$. Also, the inequality in Corollary 3.3 can be strict. For the star $K_{1,n-1}(n \ge 3)$, by Theorem 2.10(iv), $dn_{\triangle f_x}(K_{1,n-1}) = n-2 < n - \frac{D_{\triangle f}}{2}$ for an end vertex x in $K_{1,n-1}$.*

Theorem 3.5. *Let G be a connected graph of order $n \ge 2$ and $G \ne K_3$. Then $G = K_{1,n-1}$ if and only if $dn_{\triangle f_x}(G) = n - 1$ or $dn_{\triangle f_x}(G) = n - 2$ for every vertex x of G.*

Proof. If $G = K_{1,n-1}$, then by Theorem 2.10(*iv*), $dn_{\triangle f_x}(G) = n - 1$ or $dn_{\triangle f_x}(G) = n-2$ for every vertex x of G. If $n = 2$, then $G = K_2 = K_{1,n-1}$. If $n = 3$, then $G = P_3 = K_{1,n-1}$. Let $n \ge 4$. We prove that G is a star. Suppose G is not a star. If G is a tree, then G has at most $n - 2$ end-vertices. By Corollary 2.7, $dn_{\triangle f_x}(G) \le n-3$ if x is an end-vertex, which is a contradiction. Now, if G is not a tree. Let $c(G)$ be the length of a longest cycle, say C in G. If $c(G) \ge 4$, then $D_{\triangle f} \ge 3$ so that $e_{\triangle f}(x) \ge 3$ for some vertex x in G. Hence by Theorem 3.1, $dn_{\triangle f_x}(G) \le n - 3$, which is a contradiction. If $c(G) = 3$, let u, v, w, u be a triangle in G. Since $n \ge 4$, there exists $x \in V(G) - \{u, v, w\}$ such that x is adjacent to at least one of u, v, w say $xu \in E(G)$. Then x, u, v, w is a path in G so that $e_{\triangle f}(x) \ge 3$. Then by Theorem 3.1, $dn_{\triangle f_x}(G) \le n - 3$, which is a contradiction. Thus G is a star. □

Theorem 3.6. *Let G be a connected graph of order $n \ge 5$. Then $dn_{\triangle f_x}(G) = n - 2$ or $dn_{\triangle f_x}(G) = n - 3$ for every vertex x of G if and only if G is a double star or $K_{1,n-1} + e$.*

Proof. It is straightforward to verify that if G is a double star or $K_{1,n-1} + e$, then $dn_{\triangle f_x}(G) = n - 2$ or $dn_{\triangle f_x}(G) = n - 3$ for every vertex x of G. For the converse, let G be a connected graph of order $n \ge 5$

such that $dn_{\triangle f_x}(G) = n - 2$ or $dn_{\triangle f_x}(G) = n - 3$ for every vertex x of G. If $D_{\triangle f} \leq 2$, then G is the star $K_{1,n-1}$ and so by Theorem 2.10 (iv), $dn_{\triangle f_x}(G) = n - 1$ for the cut vertex x in G, which is a contradiction. Let $D_{\triangle f} = 3$. If G is a tree, then G is a double star and the result follows from Corollary 2.7. Assume that G is not a tree. Let $c(G)$ denote the length of a longest cycle in G. Since $D_{\triangle f} = 3$, it follows that $c(G) \leq 4$. We consider two cases.

Case 1. Let $c(G) = 4$. Let $C_4 : v_1, v_2, v_3, v_4, v_1$ be a 4-cycle in G. Since $n \geq 5$ and G is connected, there exists a vertex x not on C_4 such that x is adjacent to some vertex, say v_1 of C_4. Then x, v_1, v_2, v_3, v_4 is a path of length 4 in G so that $D_{\triangle f} \geq 4$, which is a contradiction.

Case 2. Let $c(G) = 3$. If G contains two or more triangles, then $c(G) = 4$ or $D_{\triangle f} \geq 4$, which is a contradiction. Hence G contains an unique triangle $C_3 : v_1, v_2, v_3, v_1$. Now, we prove that there is exactly one vertex on C_3 of degree at least 3. If there are two or more vertices of C_3 having degree 3 or more, then $D_{\triangle f} \geq 4$, which is a contradiction. Thus exactly one vertex in C_3 has degree 3 or more. Since $D_{\triangle f} = 3$, it follows that $G = K_{1,n-1} + e$. Now, it follows from Theorem 2.4 and Theorem 2.10 that $dn_{\triangle f_x}(G) = n - 2$ or $dn_{\triangle f_x}(G) = n - 3$ according as x is a cut vertex or not. If $D_{\triangle f} \geq 4$, then $e_{\triangle f}(x) \geq 4$ for some vertex x in G. Hence by Theorem 3.1, $dn_{\triangle f_x}(G) \leq n - e_{\triangle f}(x) \leq n - 4$, which is a contradiction. □

Remark 3.7. *Theorem 3.6 is not true for $n = 4$. For the graph C_4, $n = 4$ and $dn_{\triangle f_x}(G) = 1 = n - 3$ for every vertex x in G. However, G is neither a double star nor $K_{1,n-1} + e$.*

Theorem 3.8. *For every tree T with triangle free detour diameter $D_{\triangle f}$, $dn_{\triangle f_x}(G) = n - D_{\triangle f}$ or $dn_{\triangle f_x}(G) = n - D_{\triangle f} + 1$ for every vertex x of T if and only if T is a caterpillar.*

Proof. If T be any tree. Let $P : u = u_0, v_1, ..., v_{D_{\triangle f}} = v$ be a triangle free detour diametral path. Let k be the number of end vertices of T and l be the number of internal vertices of T other than $v_1, v_2, ..., v_{D_{\triangle f}-1}$. Then $D_{\triangle f} - 1 + l + k = n$. By Corollary 2.6, $dn_{\triangle f_x}(T) = k$ or $dn_{\triangle f_x}(T) = k - 1$ for every vertex x of T and so $dn_{\triangle f_x}(T) = n - D_{\triangle f} - 1 + l$ or $dn_{\triangle f_x}(T) = n - D_{\triangle f} - 1$ for every vertex x of T. Hence $dn_{\triangle f_x}(T) = n - D_{\triangle f} + l$ or $dn_{\triangle f_x}(T) = n - D_{\triangle f}$ for every vertex x of T if and only if $l = 0$ if and only if all the internal vertices of T lie on the triangle free detour diametral path P if and only if T is a caterpillar. □

Theorem 3.9. *For every pair a, b of integers with $1 \leq a \leq b$, there exists a connected graph G with $d_x(G) = a$ and $dn_{\triangle f_x}(G) = b$.*

Proof. **Case 1.** For $1 \leq a = b$, any tree with a end vertices has the desired properties, by Theorem 2.5 and Corollary 2.7.

Case 2. For $1 \leq a < b$. Let $P_i : v_i (1 \leq i \leq b - a)$ be a $b - a$ copies of a path of order 1 and $P : x, u_1, u_2, u_3$ a path of order 4. Let G be the graph obtained by joining each $v_i (1 \leq i \leq b - a)$ in P_i and u_1 in P and u_2 in P. Adding a new vertices $w_1, w_2, ..., w_a$ and joining each $w_i (1 \leq i \leq a)$ to u_3. The resulting graph G of order $b + 4$ is shown in Figure 3.2. Let $S_1 = \{x, w_1, w_2, ..., w_a\}$ be the set of all extreme vertices of G. It is easily verified that $S = S_1 - \{x\}$ is a x-detour set of G and so by Theorem 1.4, $d_x(G) = |S| = a$.

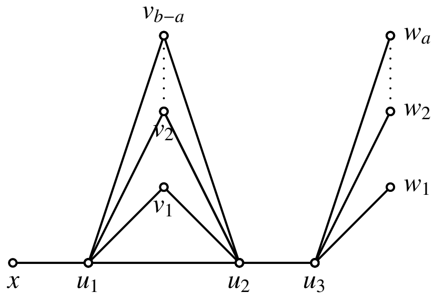

Figure 3.2: G

Next, we show that $dn_{\triangle f_x}(G) = b$. By Theorem 2.4, every x-triangle free detour set of G contains S. Clearly, S is not a triangle free detour set of G. It is easily verified that each $v_i (1 \leq i \leq b - a)$ must belong to every x-triangle free detour set of G. Thus $T = S \cup \{v_1, v_2, ..., v_{b-a}\}$ is a x-triangle free detour set of G, it follows from Theorem 2.4 that T is a x-triangle free detour basis of G and so $dn_{\triangle f_x}(G) = b$.

□

Theorem 3.10. *For every pair a, b of integers with $1 \leq a \leq b$, there exists a connected graph G with $dn_{\triangle f_x}(G) = a$ and $dm_x(G) = b$.*

Proof. **Case 1.** For $1 \leq a = b$, any tree with a end vertices has the desired properties, by Theorem 2.4 and Corollary 2.6.

Case 2. For $1 \leq a < b$. Let $P_i : s_i, t_i (1 \leq i \leq b - a)$ be a $b - a$ copies of a path of order 2 and $P : x, u_1, u_2, u_3$ a path of order 4. Let G be the graph obtained by joining each $s_i (1 \leq i \leq b - a)$ in P_i to u_1 in P and joining each $t_i (1 \leq i \leq b-a)$ in P_i to u_2 in P. Adding a new vertices $w_1, w_2, ..., w_a$ and joining each $w_i (1 \leq i \leq a)$ to u_3. The resulting graph G of order $2b - a + 4$ is shown in Figure 3.3. Let $S_1 = \{x, w_1, w_2, ..., w_a\}$ be the set of all extreme vertices of G. It is easily verified that $S = S_1 - \{x\}$ is a x-trianlge free detour set of G and so by Theorem 2.4, $dn_{\triangle f_x}(G) = |S| = a$.

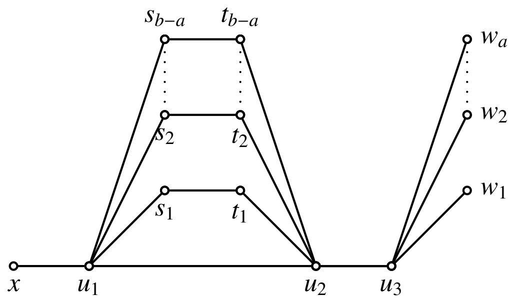

Figure 3.3: G

Next, we show that $dm_x(G) = b$. By Theorem 1.5, every x-detour monophonic set of G contains S. Clearly, S is not a detour monophonic set of G. It is easily verified that each $s_i(1 \leq i \leq b-a)$ or each $t_i(1 \leq i \leq b-a)$ must belong to every x-detour monophonic set of G. Thus $T = S \cup \{s_1, s_2, ..., s_{b-a}\}$ is a x-detour monophonic set of G, it follows from Theorem 2.5 that T is a x-detour monophonic basis of G and so $dm_x(G) = b$.

□

Theorem 3.11. *For every pair a, b of integers with* $1 \leq a \leq b$, *there exists a connected graph G with* $dn_{\triangle f_x}(G) = a$ *and* $g_x(G) = b$.

Proof. This follows from Theorem 3.10. □

Theorem 3.12. *For positive integers a, b and* $c \geq 2$ *with* $a < b$, *there exists a connected graph G with* $R_{\triangle f}(G) = a$, $D_{\triangle f}(G) = b$ *and* $dn_{\triangle f_x}(G) = c$ *or* $dn_{\triangle f_x}(G) = c - 1$ *for every vertex x of G.*

Proof. If $a = 1$, then $b = 2$. Take $G = K_{1,c}$. Then by Theorem 2.9(iv), $dn_{\triangle f_x}(G) = c$ or $dn_{\triangle f_x}(G) = c - 1$ for every vertex x of G. Now, let $a \geq 2$. We construct a graph G with the desired properties as follows.

Let $C_{a+1} : v_1, v_2, ..., v_{a+1}, v_1$ be a cycle of order $a + 1$ and let $P_{b-a+1} : u_0, u_1, ..., u_{b-a}$ be a path of order $b - a + 1$. Let H be a graph obtained from C_{a+1} and P_{b-a+1} by identifying v_1 in C_{a+1} and u_0 in P_{b-a+1}. Now, add $c-2$ new vertices $w_1, w_2, ..., w_{c-2}$ to H by joining each vertex $w_i(1 \leq i \leq c-2)$ to the vertex u_{b-a-1} and obtain the graph G of Figure 3.4. Now, $R_{\triangle f} = a$, $D_{\triangle f} = b$ and G has $c - 1$ end vertices.

Case 1. Let a be even. If $a = 2$, then $dn_{\triangle f_x}(G) = c$ or $dn_{\triangle f_x}(G) = c - 1$ according as $x \in \{v_1, u_1, u_2, ..., u_{b-a-1}\}$ or $x \in \{v_2, v_3, u_{b-a}, w_1, w_2, ..., w_{c-2}\}$. If $a \geq 4$, then $dn_{\triangle f_x}(G) = c$ or $dn_{\triangle f_x}(G) = c - 1$ according as $x \in \{v_1, v_3, v_4, ..., v_a, u_1, u_2, ..., u_{b-a-1}\}$ or $x \in \{v_2, v_{a+1}, u_{b-a}, w_1, w_2, .., w_{c-2}\}$.

Case 2. Let a be odd. If $a = 3$, then $dn_{\triangle f_x}(G) = c$ or $dn_{\triangle f_x}(G) = c - 1$ according as $x \in \{v_1, u_1, u_2, ..., u_{b-a-1}\}$ or $x \in \{v_2, v_3, v_4, u_{b-a}, w_1, w_2, ..., w_{c-2}\}$. If $a \geq 5$, then $dn_{\triangle f_x}(G) = c$ or $dn_{\triangle f_x}(G) = c - 1$ according as $x \in \{v_1, v_3, v_4, ..., v_{(a+1)/2}, v_{(a+5)/2}, ..., v_a, u_1, u_2, ..., u_{b-a-1}\}$ or $x \in \{v_2, v_{(a+3)/2}, v_{a+1}, u_{b-a}, w_1, w_2, ..., w_{c-2}\}$. Thus $dn_{\triangle f_x}(G) = c$ or $dn_{\triangle f_x}(G) = c - 1$ for every vertex x of G.

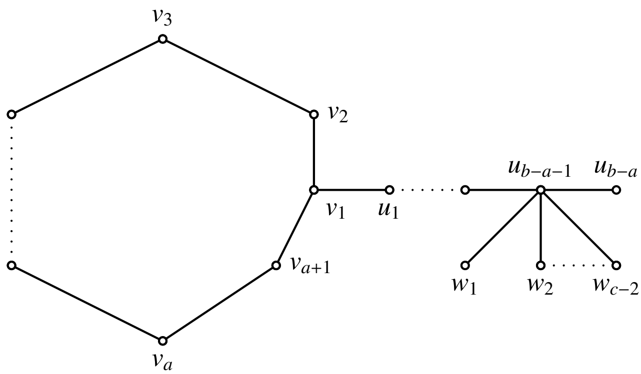

Figure 3.4: G

□

Theorem 3.13. *For each triple a, b and n of positive integers with* $1 \leq b \leq n - a + 1$ *and* $a \geq 4$*, there exists a connected graph G of order n with triangle free detour diameter* $D_{\triangle f} = a$ *and* $dn_{\triangle f_x}(G) = b$ *or* $dn_{\triangle f_x}(G) = b - 1$ *for every vertex x of G.*

Proof. Let G be a graph obtained from the cycle $C_a : u_1, u_2, ..., u_a, u_1$ of order a by (i) adding $b - 1$ new vertices $v_1, v_2, ..., v_{b-1}$ and joining each vertex $v_i (1 \leq i \leq b - 1)$ to u_1 and (ii) adding $n - a - b + 1$ new vertices $w_1, w_2, ..., w_{n-a-b+1}$ and joining each vertex $w_i (1 \leq i \leq n - a - b + 1)$ to both u_1 and u_3. The graph G has order n and triangle free detour diameter a and is shown in Figure 3.5. If $b = 1$, $dn_{\triangle f_x}(G) = b$ for every vertex x in G. If $b \geq 2$, then we consider two cases.

Case 1. Let a be even. If $a = 4$, then $dn_{\triangle f_x}(G) = b$ or $dn_{\triangle f_x}(G) = b - 1$ according as $x = u_1$ or $x \in \{u_2, u_3, u_4, v_1, v_2, ..., v_{b-1}, w_1, w_2, ..., w_{n-a-b+1}\}$. If $a \geq 6$, then $dn_{\triangle f_x}(G) = b$ or $dn_{\triangle f_x}(G) = b - 1$ according as $x \in \{u_1, u_2, ..., u_{a/2}, u_{(a+4)/2}, ..., u_{a-1}, w_1, ..., w_{c-a-b+1}\}$ or $x \in \{u_{(a+2)/2}, u_a, v_1, v_2, v_{b-1}\}$.

Case 2. Let a be odd. Clearly $dn_{\triangle f_x}(G) = b$ or $dn_{\triangle f_x}(G) = b-1$ according as $x \in \{u_1, u_2, u_2, ..., u_{a-1}, w_1, ..., w_{n-a-b+1}\}$ or $x \in \{u_a, v_1, v_2, ..., v_{b-1}\}$. Thus $dn_{\triangle f_x}(G) = b$ or $dn_{\triangle f_x}(G) = b - 1$ for every vertex x of G. □

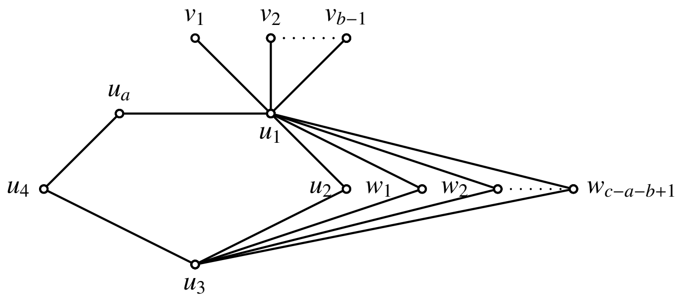

Figure 3.5: G

Theorem 3.14. *Let $n \geq 2$ be any integer. For $1 \leq a \leq n-1$, there exists a connected graph G with order n and $dn_{\triangle f_x}(G) = a$ or $dn_{\triangle f_x}(G) = a-1$ for every vertex x of G.*

Proof. For $n = 2$, $G = K_2$ has the desired properties. For $n = 3$, $G = C_3$ or P_3 has the desired properties according as $a = 1$ or $a = 2$. For $n \geq 4$, we consider three cases.

Case 1. Let $a = 1$. Then $G = C_n$ has the desired properties.

Case 2. Let $2 \leq a \leq n-2$. Then $n-a+1 \geq 3$. The graph G is obtained from the cycle $C_{n-a+1} : u_1, u_2, ..., u_{n-a+1}, u_1$ by adding the $a-1$ new vertices $v_1, v_2, ..., v_{a-1}$ and joining these to u_1. The graph G is shown in Figure 3.6.

Subcase a. Let $n-a+1$ be even. If $n-a+1 = 4$, then $dn_{\triangle f_x}(G) = a$ or $dn_{\triangle f_x}(G) = a-1$ according as $x = u_1$ or $x \in \{u_2, u_3, u_4, v_1, v_2, ..., v_{a-1}\}$. If $n-a+1 \geq 6$, then $dn_{\triangle f_x}(G) = a$ or $dn_{\triangle f_x}(G) = a-1$ according as $x \in \{u_1, u_3, u_4, ..., u_{(n-a+1)/2}, u_{(n-a+5)/2}, ..., u_{n-a}\}$ or $x \in \{u_2, u_{(n-a+3)/2}, u_{n-a+1}, v_1, v_2, ..., v_{a-1}\}$.

Subcase b. Let $n-a+1$ be odd. If $n-a+1 = 3$, then $dn_{\triangle f_x}(G) = a$ or $dn_{\triangle f_x}(G) = a-1$ according as $x = u_1$ or $x \in \{u_2, u_3, v_1, v_2, ..., v_{a-1}\}$. If $n-a+1 \geq 5$, then $dn_{\triangle f_x}(G) = a$ or $dn_{\triangle f_x}(G) = a-1$ according as $x \in \{u_1, u_3, u_4, ..., u_{n-a}\}$ or $x \in \{u_2, u_{n-a+1}, v_1, v_2, ..., v_{a-1}\}$. Thus $dn_{\triangle f_x}(G) = a$ or $dn_{\triangle f_x}(G) = a-1$ for every vertex x of G.

Case 3. Let $a = n-1$. Then $G = K_{1,n-1}$ has the desired properties. □

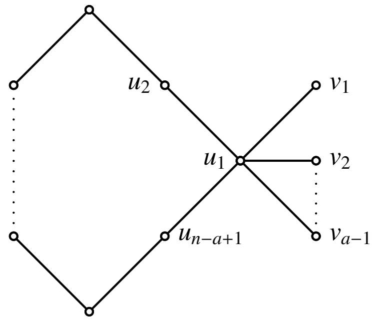

Figure 3.6: G

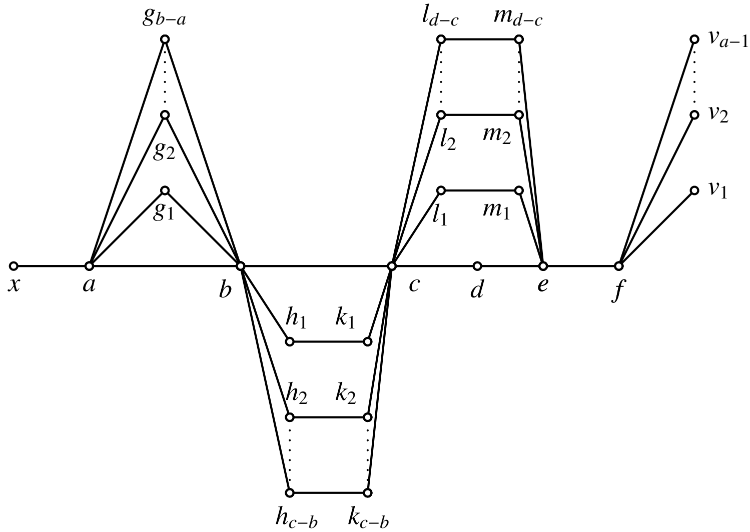

Figure 3.7: G

Theorem 3.15. *For any four positive integers a, b, c and d of with $2 \le a \le b \le c \le d$, there exists a connected graph G such that $d_x(G) = a$, $dn_{\triangle f_x}(G) = b$, $dm_x(G) = c$ and $g_x(G) = d$.*

Proof. Let $2 \le a \le b \le c \le d$. Let $P : x, a, b, c, d, e, f$ be a path of order 7 and adding $a-1$ new vertices $v_1, v_2, v_3, v_4, \ldots.., v_{a-1}$ to f. Let $P_i : g_i(1 \le i \le b-a)$ be a $b-a$ copies of K_1 and joining each $g_i(1 \le i \le b-a)$ in P_i to a and b in P. Let $P_j : h_j, k_j(1 \le j \le c-b)$ be a $c-b$ copies of a path of length 2 and joining each $h_j(1 \le j \le c-b)$ in P_j to b in P and joining each $k_j(1 \le j \le c-b)$ in P_j to c in P. Let $P_k : l_k, m_k(1 \le k \le d-c)$ be a

$d - c$ copies of a path of order 2 and joining each $l_k (1 \leq k \leq d - c)$ in P_k to c in P and joining $m_k (1 \leq k \leq d - c)$ in P_k to e in P. The resulting graph G is shown in Figure 3.7.

It is easily verify that $S_1 = \{d, v_1, v_2, ..., v_{a-1}\}$ is a minimum x-detour set, $S_2 = S_1 \cup \{g_1, g_1, g_2, ..., g_{b-a}\}$ is a x-triangle free detour basis, $S_3 = S_2 \cup \{h_1, h_2, h_3, ..., h_{c-b}\}$ is a minimum x-detour monophonic set and $S_4 = S_3 \cup \{l_1, l_2, l_2, ..., l_{d-c}\}$ is a minimum x-geodetic set. Thus $dn_x(G) = a$, $dn_{\triangle f_x}(G) = b$, $dm_x(G) = c$ and $g_x(G) = d$. □

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