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The k-Local Colouring of Jahangir Graphs and Some Characteristics of the k-Locally Rainbow Graphs

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Abstract

The k-local chromatic number of the Jahangir graphs and some characteristics of the k-locally rainbow graphs are studied in this article.

Keywords: k-local colouring, k-local chromatic number, Jahangir graph, k-local rainbow colouring, k-locally rainbow graphs, k-locally nearly rainbow graphs

Mathematics Subject Classification (2010): 05C10

1. Introduction

Graphs considered in this paper are undirected connected and simple graphs on n vertices. For standard notations and terminologies we follow Harary.[8] A *proper vertex colouring* of a graph is an assignment of colours to its vertices so that no two adjacent vertices have the same colour. The *chromatic number* $\chi(G)$ is defined as the minimum number of colours used in any colouring of G. A *k-colouring* of G uses k colours. The *value* of a colouring c of G is defined by $\chi(c) = \max\{c(v) : v \in V(G)\}$. Then $\chi(G) = \min\{\chi(c) : c$ is a colouring of $G\}$. The generalisations of graph colouring have been introduced and the variations are developed. The area of research in graph colouring is branching out in many directions. Chartrand *et al.* have introduced the study of local colourings of graphs.[7] The definition of graph colouring is generalised in the definition of k-local colouring.[1, 2, 5] For a graph G on n vertices, let $S \subseteq V(G)$, and m_s be the size of the induced subgraph $\langle S \rangle$ of G.

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A *k-local colouring* of a graph G of order $n \geq 2$ and $2 \leq k \leq n$ is a function $c : V \to \mathbb{N}$ such that for each subset $S \subseteq V(G)$ with $2 \leq |S| \leq k$, there exists two distinct vertices $u, v \in S$ such that $|c(u) - c(v)| \geq m_s$, where m_s is the size of the induced subgraph $\langle S \rangle$ of G. The *value of a k-local colouring c* is the maximum colour it assigns to a vertex of G and is denoted by $lc_k(c)$. The *k-local chromatic number* of G is the minimum value of any k-local colouring of the graph G and is denoted by $lc_k(G)$. The k-local colouring of G is the generalisation of the colouring of G, since the condition on colours that can be assigned to the vertices of G depends on subgraphs of order k, where $2 \leq k \leq n$ rather than only on subgraphs of order 2.
A 3-local colouring c of a graph G is referred to as *local colouring of G*. $lc_3(c)$ is denoted as $\chi_l(c)$. If $\chi_l(c) = \chi_l(G)$, then c is called a minimum local colouring of G and the value of the minimum local colouring is called as the local chromatic number of G.

2. Preliminaries

The k-local colouring of a graph G is its colouring if k=2. The k-local chromatic number of a graph is 1 if and only if it is a totally disconnected graph. For every integer k with $2 \leq k \leq n$, it follows that $\chi(G) \leq lc_k(G)$ and $lc_{k-1}(G) \leq lc_k(G)$. A k-local colouring of a graph with $lc_k(G) = r$ need not use all r colours, but the colours 1 and r must be assigned at least once. If H is a subgraph of G, then $lc_k(H) \leq lc_k(G)$. Chartrand *et al.* have stated the result for complementary colouring for 3-local colouring of graphs.[6] We have extended it for k-local colouring in [3].

3. The k-Local Chromatic Number of Jahangir Graphs

The *Jahangir graph* $J_{n,m}$ for $m \geq 3$, $n \geq 2$ is a graph on $nm + 1$ vertices consisting of a cycle C_{nm} with one additional vertex at the centre which is adjacent to m vertices of C_{nm} at a distance n to each other on C_{nm}.

Note 3.1. *Let $J_{n,m}$ be a Jahangir graph with $nm + 1$ vertices and $nm + m$ edges. Let the vertex set be $V = \{v_0, v_{1,0}, v_{1,1}, v_{1,2}, \cdots, v_{1,n-1}, v_{2,0}, v_{2,1}, \cdots, v_{2,n-1}, \cdots, v_{m,0}, v_{m,1}, \cdots, v_{m,n-1}\}$ where v_0 is the vertex at the centre which is adjacent to the m vertices $\{v_{1,0}, v_{2,0} \cdots, v_{m,0}\}$ and the edge set be $E = \{v_0 v_{i,0}, v_{i,j} v_{i,j+1} / 1 \leq i \leq m, 0 \leq j \leq n-2\} \cup \{v_{i,n-1}, v_{i+1,0} / 1 \leq i \leq m-1\} \cup \{v_{m-1,n-1} v_{1,0}\}$.*

Hereafter we will use the above notation.

Lemma 3.2. *The k-local chromatic number of the Jahangir graph $J_{n,m}$, that is, $lc_k(J_{n,m}) \geq k + \lfloor\frac{k-2}{n}\rfloor$ where k is any integer such that $2 \leq k \leq |V(J_{n,m})|$.*

Proof. Let $J_{n,m}$ be a Jahangir graph. Let S be a subset of $V(J_{n,m})$ with k vertices, where $2 \leq k \leq nm + 1$.
Case 1: Let $v_0 \in S$. If $S = \{v_0, v_{1,0}, v_{1,1}, v_{1,2}, \cdots, v_{1,n-1}, v_{2,0}, v_{2,1}, \cdots, v_{i,j}\}$, where $i = \lfloor\frac{k-2}{n}\rfloor + 1$ and $(k-2) \equiv j \mod n$. Except v_0 the remaining $k-1$ vertices form a path P_{k-1}. Here the i vertices $v_{1,0}, v_{2,0}, \cdots, v_{i,0}$ are adjacent to v_0 and hence there are i edges. The path P_{k-1} has $(k-2)$ edges. Therefore, $m_s = i + (k-2) = \lfloor\frac{k-2}{n}\rfloor + 1 + k - 2$. Hence $m_s = \lfloor\frac{k-2}{n}\rfloor + k - 1$.
Case 2: Let $v_0 \notin S$. Suppose $S = \{v_{1,0}, v_{1,1}, v_{1,2}, \cdots, v_{1,n-1}, v_{2,0}, v_{2,1}, \cdots, v_{i,j}\}$ where $i = \lfloor\frac{k-1}{n}\rfloor + 1$, $1 \leq i \leq m$ and $(k-1) \equiv j \mod n$. The induced subgraph $\langle S \rangle$ of G is a path. Hence $m_s = k-1$. Hence in both the cases $m_s \leq \lfloor\frac{k-2}{n}\rfloor + k - 1$.
Similarly we can prove that, for any other k-subset S of $V(J_{n,m})$, $m_s \leq \lfloor\frac{k-2}{n}\rfloor + k - 1$. Hence $\lfloor\frac{k-2}{n}\rfloor + k - 1$ is the maximum value of m_s for all k-subset S of $V(J_{n,m})$. Therefore, $lc_k(J_{n,m}) \geq m_s + 1 = \lfloor\frac{k-2}{n}\rfloor + k$. Hence $lc_k(J_{n,m}) \geq \lfloor\frac{k-2}{n}\rfloor + k$. □

Lemma 3.3. *For the Jahangir graph $J_{n,m}$, $lc_k(J_{n,m}) \leq \lfloor\frac{k-2}{n}\rfloor + k$, where k is any integer such that $2 \leq k \leq |V(J_{n,m})|$ and n is even.*

Proof. Let $J_{n,m}$ be a Jahangir graph. By Lemma 3.2, the maximum value of m_s for any k-subset S of $V(J_{n,m})$ is, $\lfloor\frac{k-2}{n}\rfloor + k - 1$. Let $r = m_s + 1 = \lfloor\frac{k-2}{n}\rfloor + k$ Define $c : V \to \mathbb{N}$ as follows: When n is even, $c(v_0) = 1$, for $1 \leq i \leq m$ and $1 \leq j \leq n$

$$c(v_{i,j}) = \begin{cases} 1, & \text{if } j \equiv 1 \ (mod\ 2) \\ r, & \text{if } j \equiv 0 \ (mod\ 2) \end{cases}$$

Suppose S contains the vertex v_0 and the vertex $v_{i,j}$ coloured with r then $|c(v_0) - c(v_{i,j})| = |1 - r| = r - 1 \geq m_s$. Suppose S contains the vertex v_0 and all the other vertices of S are coloured as 1, then $m_s = 0$ as the induced subgraph $\langle S \rangle$ of G, with the vertex v_0 and the vertices coloured by 1 is totally disconnected and $|c(v_0) - c(v_{i,j})| = 0 = m_s$. Suppose $v_0 \notin S$ and $v_{i,j}, v_{i,j+1} \in S$ for some i, then $|c(v_{i,j}) - c(v_{i,j+1})| = r - 1 \geq m_s$. Suppose $v_0 \notin S$ and no such vertices that are adjacent like $v_{i,j}, v_{i,j+1}$ exist in S then $m_s = 0$. Then for any two vertices $v_{i,j}, v_{k,l} \in S$, $|c(v_{k,l}) - c(v_{i,j})| = 0 = m_s$. Hence the k-local colour condition is satisfied for any subset $S \subseteq V(J_{n,m})$ where $|S| = k$. Hence c is a k-local colouring. Therefore, $lc_k(J_{n.m}) \leq r$ when n is even. □

By Lemma 3.2 and Lemma 3.3, we have

Theorem 3.4. *The k-local chromatic number of the Jahangir graph $J_{n,m}$ is $k + \lfloor\frac{k-2}{n}\rfloor$ when n is even and k is any integer such that $2 \leq k \leq |V(J_{n,m})| = nm + 1$.*

Lemma 3.5. *For the Jahangir graph $J_{n,m}$, $lc_k(J_{n,m}) \leq \lfloor \frac{k-2}{n} \rfloor + k$, where k is any odd integer such that $3 \leq k \leq |V(J_{n,m})|$ and n is odd.*

Proof. Let $J_{n,m}$ be a Jahangir graph. By Lemma 3.2, the maximum value of m_s for any k-subset S of $V(J_{n,m})$ is, $\lfloor \frac{k-2}{n} \rfloor + k - 1$. Let $r = m_s + 1 = \lfloor \frac{k-2}{n} \rfloor + k$. Let n and k be odd. Define $c : V \to \mathbb{N}$ as follows: $c(v_0) = 1$, for $1 \leq i \leq m$ and $1 \leq j \leq n$

$$c(v_{i,j}) = \begin{cases} 1, \text{ if } j \equiv 1 \ (mod\ 2) \\ r, \text{ if } j \equiv 0 \ (mod\ 2) \text{ and } j \neq n-1 \\ \lceil \frac{r}{2} \rceil \text{ if } j = n-1 \end{cases}$$

As discussed in Lemma 3.3, when the k-subset S contains the vertices coloured as 1 and r, the k-local colour condition is satisfied whether $v_0 \in S$ or $v_0 \notin S$. Suppose S contains either the vertices coloured as 1 and $\lceil \frac{r}{2} \rceil$ only, or the vertices coloured as r and $\lceil \frac{r}{2} \rceil$ only, then in both the cases $m_s \leq \lfloor \frac{k}{2} \rfloor = \frac{k-1}{2}$ as k is odd. We discuss the following cases.
Case 1: When S contains the vertices coloured as 1 and $\lceil \frac{r}{2} \rceil$ only. Let $v_{i,j}$ and $v_{s,t} \in S$ be the two vertices with the colours as 1 and $\lceil \frac{r}{2} \rceil$ respectively. Then $|c(v_{s,t}) - c(v_{i,j})| = ||\lceil \frac{r}{2} \rceil - 1| = \lceil \frac{k + \lfloor \frac{k-2}{n} \rfloor}{2} \rceil - 1 \geq m_s$
Case 2: When S contains the vertices coloured as r and $\lceil \frac{r}{2} \rceil$ only. Let $v_{i,j}$ and $v_{s,t} \in S$ be the two vertices with the colours r and $\lceil \frac{r}{2} \rceil$ respectively. $|c(v_{i,j}) - c(v_{s,t})| = |r - \lceil \frac{r}{2} \rceil| > m_s = \frac{k-1}{2}$. Hence when k is odd the k-local colour condition is satisfied.
Hence the k-local colour condition is satisfied for any k-subset $S \subseteq V(J_{n,m})$, when k is odd. Hence c is a k-local colouring when k is odd and n is odd. Therefore, $lc_k(J_{n.m}) \leq r$ when k and n are odd where $r = k + \lfloor \frac{k-2}{n} \rfloor$. □

By Lemma 3.2 and Lemma 3.5 we have

Theorem 3.6. *The k-local chromatic number of the Jahangir graph $J_{n,m}$ is $k + \lfloor \frac{k-2}{n} \rfloor$ when n is odd and k is any odd integer such that $3 \leq k \leq |V(J_{n,m})| = nm + 1$.*

Lemma 3.7. *For the Jahangir graph $J_{n,m}$, when n is odd, $lc_k(J_{n,m}) \leq r = \lfloor \frac{k-2}{n} \rfloor + k$, where k is any even integer such that $2 \leq k \leq |V(J_{n,m})|$ with the conditions either (i) $m < r - 1$ or (ii) when $m > r - 1$, $\lfloor \frac{k-2}{n} \rfloor \geq 1$.*

Proof. Let $J_{n,m}$ be a Jahangir graph. By Lemma 3.2, the maximum value of m_s for any k-subset S of $V(J_{n,m})$ is, $\lfloor \frac{k-2}{n} \rfloor + k - 1$. Let $r = m_s + 1 = \lfloor \frac{k-2}{n} \rfloor + k$. Let n be odd and k be even. Define the k-local colouring c as in Lemma 3.5 to the Jahangir graph $J_{n,m}$. As discussed in Lemma 3.3, when the k-subset S contains the vertices coloured as 1 and r, the k-local colour condition is satisfied whether $v_0 \in S$ or $v_0 \notin S$.
Case 1: Let $\frac{k}{2} < m \leq r - 1$. Redefine the colouring of the vertices

as follows: $c(v_{i,j}) = i+1$ if $j = n-1, 1 \leq i \leq m$ and the colouring of the other vertices be the same. If $m \geq \frac{k}{2}$, then we can choose the subset S that contains k vertices that are coloured as either 1 and $i+1$ only, or the vertices coloured as r and $i+1$ only then in both the cases $m_s \leq \lfloor \frac{k}{2} \rfloor = \frac{k}{2}$ as k is even. We discuss the following cases.
Sub case i:
When S contains the vertices coloured as 1 and $i+1$ only. Here $m_s \leq \frac{k}{2}$. There exists an $i \geq \frac{k}{2}$, such that $v_{i,n-1} \in S$. Let $v_{i,n-1}$ and $v_{s,t} \in S$ be the two vertices with the colours as $i+1$ and 1 respectively. $|c(v_{s,t}) - c(v_{i,n-1})| = |1-(i+1)| = i \geq \frac{k}{2} = m_s$. Hence the k-local colour condition is satisfied.
Sub case ii: When S contains the vertices coloured as r and $i+1$ only. There exists an $i \leq \frac{k}{2} - 1$, such that $v_{i,n-1} \in S$. Let $v_{i,n-1}$ and $v_{s,t} \in S$ be the two vertices with the colours $i+1$ and r respectively. $|c(v_{i,n-1}) - c(v_{s,t})| = |i+1-r| = r-i-1 = k+\lfloor \frac{k-2}{n} \rfloor - i - 1 \geq \frac{k}{2} + 1 + \lfloor \frac{k-2}{n} \rfloor - 1 \geq \frac{k}{2} \geq m_s$.
Case 2: Let $m \geq r-1$ and $\lfloor \frac{k-2}{n} \rfloor \geq 1$. If $m \geq r-1$, then we can choose the subset S that contains k vertices that are coloured as either 1 and $\lceil \frac{r}{2} \rceil$ only, or the vertices coloured as r and $\lceil \frac{r}{2} \rceil$ only then in both the cases $m_s \leq \lfloor \frac{k}{2} \rfloor = \frac{k}{2}$ as k is even. We discuss the following sub cases.
Sub case i: When S contains the vertices coloured as 1 and $\lceil \frac{r}{2} \rceil$ only. Here $m_s = \frac{k}{2}$. Let $v_{i,j}$ and $v_{s,t} \in S$ be the two vertices with the colours as 1 and $\lceil \frac{r}{2} \rceil$ respectively. $|c(v_{s,t}) - c(v_{i,j})| = |\lceil \frac{r}{2} \rceil - 1| = \lceil \frac{k + \lfloor \frac{k-2}{n} \rfloor}{2} \rceil - 1 \geq m_s$, as $\lfloor \frac{k-2}{n} \rfloor \geq 1$. Hence the k-local colour condition is satisfied.
Sub case ii: When S contains the vertices coloured as r and $\lceil \frac{r}{2} \rceil$ only. Let $v_{i,j}$ and $v_{s,t} \in S$ be the two vertices with the colours r and $\lceil \frac{r}{2} \rceil$ respectively. $|c(v_{i,j}) - c(v_{s,t})| = |r - \lceil \frac{r}{2} \rceil| = k + \lfloor \frac{k-2}{n} \rfloor - \lceil \frac{k + \lfloor \frac{k-2}{n} \rfloor}{2} \rceil \geq \frac{k}{2} = m_s$.
Case 3: When $m < \frac{k}{2}$.The two cases discussed in Case 1 and Case 2 do not arise here. The k-local colour condition is satisfied for any k-subset of S.
Hence when k is even and $m < r-1$ the k-local colour condition is satisfied. When k is even, $m \geq r-1$ and $\lfloor \frac{k-2}{n} \rfloor \geq 1$, the k-local colour condition is satisfied.
Hence the k-local colour condition is satisfied for any k-subset $S \subseteq V(J_{n,m})$, when k is even with the conditions either (i) $m < r-1$ or (ii) when $m \geq r-1$, $\lfloor \frac{k-2}{n} \rfloor \geq 1$. Hence c is a k-local colouring when k is even and n is odd with the conditions either (i) $m < r-1$ or (ii) when $m \geq r-1$, $\lfloor \frac{k-2}{n} \rfloor \geq 1$. Therefore, $lc_k(J_{n.m}) \leq r$ when n is odd and k is even with the conditions either (i) $m<r-1$ or (ii) when $m \geq r-1$, $\lfloor \frac{k-2}{n} \rfloor \geq 1$. □

By Lemma 3.2 and Lemma 3.7, we have

Theorem 3.8. *The k-local chromatic number of the Jahangir graph $J_{n,m}$ is $k + \lfloor \frac{k-2}{n} \rfloor$ when n is odd and k is even with the conditions (i) $m < r-1$*

and (ii) when $m \geq r-1$, $\lfloor \frac{k-2}{n} \rfloor \geq 1$.

Remark 3.9. *The k-local chromatic number of the Jahangir graph is to be obtained when n is odd, k is even, $m > r-1$ and $\lfloor \frac{k-2}{n} \rfloor = 0$. When $\lfloor \frac{k-2}{n} \rfloor = 0$, $m > k$, as $r = k + \lfloor \frac{k-2}{n} \rfloor$. Hence the k-local chromatic number of the Jahangir graph to be determined when n is odd, k is even, $\lfloor \frac{k-2}{n} \rfloor = 0$ and $m > k$.*

Lemma 3.10. *The k-local chromatic number of the Jahangir graph $J_{n,m}$ is less than or equal to $k+1$ when n is odd, k is even, $k < m$ and $\lfloor \frac{k-2}{n} \rfloor = 0$.*

Proof. Let n be odd and k be even. Let $J_{n,m}$ be a Jahangir graph with $k < m$ and $\lfloor \frac{k-2}{n} \rfloor = 0$. Let $r = k+1+\lfloor \frac{k-2}{n} \rfloor = k+1$. Define the colouring of the Jahangir graph as in Lemma 3.5 with the above value $r = k+1$. As discussed in Lemma 3.3, when the k-subset S contains the vertices coloured as 1 and r, the k-local colour condition is satisfied whether $v_0 \in S$ or $v_0 \notin S$.
Case 1: When S contains the vertices coloured as 1 and $\lceil \frac{r}{2} \rceil$ only. Here $m_s \leq \frac{k}{2}$. Let $v_{i,j}$ and $v_{s,t} \in S$ be the two vertices with the colours as 1 and $\lceil \frac{r}{2} \rceil$ respectively. $|c(v_{s,t}) - c(v_{i,j})| = ||\lceil \frac{r}{2} \rceil - 1| = \lceil \frac{k+1}{2} \rceil - 1 = \frac{k+2}{2} - 1$, (as k is even) $= \frac{k}{2} \geq m_s$.
Case 2: When S contains the vertices coloured as r and $\lceil \frac{r}{2} \rceil$ only. $m_s \leq \frac{k}{2}$. Let $v_{i,j}$ and $v_{s,t} \in S$ be the two vertices with the colours r and $\lceil \frac{r}{2} \rceil$ respectively. $|c(v_{i,j}) - c(v_{s,t})| = |r - \lceil \frac{r}{2} \rceil| = k+1-\lceil \frac{k+1}{2} \rceil = k+1-\frac{k+2}{2} = \frac{k}{2} \geq m_s$. Hence the k-local colour condition is satisfied for any k-subset $S \subseteq V(J_{n,m})$, when n is odd, k is even, $k < m$ and $\lfloor \frac{k-2}{n} \rfloor = 0$. Hence c is a k-local colouring when n is odd, k is even, $k < m$ and $\lfloor \frac{k-2}{n} \rfloor = 0$. Therefore, $lc_k(J_{n,m}) \leq k+1$ when n is odd, k is even, $k < m$ and $\lfloor \frac{k-2}{n} \rfloor = 0$. □

By Lemma 3.2 and Lemma 3.10, we have

Theorem 3.11. *The k-local chromatic number of the Jahangir graph $J_{n,m}$ is $k \leq lc_k(J_{n,m}) \leq k+1$ when n is odd, k is even, $k < m$ and $\lfloor \frac{k-2}{n} \rfloor = 0$.*

4. Some Characteristics of k-Locally rainbow Graphs

In this section we consider the k-locally rainbow graphs. Local rainbow colouring and locally rainbow graphs are defined by Chartrand *et al.* [7] The k-local rainbow colouring, k-locally rainbow graphs and k-locally nearly rainbow graphs that are defined in [4] are given below.
Definition A: A k-local colouring c of G is defined as a *k-local rainbow colouring* if for each integer i with $1 \leq i \leq r = lc_k(G)$ there is a vertex v of G for which $c(v) = i$, that is, c uses all of the colours $1, 2, \cdots, r$.

Definition B: A graph G is defined as *k-locally rainbow* if every minimum k-local colouring c of G is a k-local rainbow colouring.
Definition C: A graph G with $lc_k(G) = \rho$ is called a *k-locally nearly rainbow* if there exists a minimal k-local colouring c of G such that for each i with $1 \leq i \leq \rho$, there is a vertex v of G for which $c(v) = i$.
That is, A graph G is called a *k-locally nearly rainbow* if there exists a minimal k-local colouring of G which is a k-local rainbow colouring.
Some important observations are stated below.

Observation 4.1. *Any odd cycle of order* > 3 *is a locally rainbow graph.*

Observation 4.2. *There exists k-locally rainbow graphs G such that* $\chi(G) = lc_k(G)$. *When* $k = 3$, $\chi(G) = lc_k(G) = \chi_l(G)$, *where G is an odd cycle of order* > 3.

Observation 4.3. *Any path* P_n *has a k-local rainbow colouring. The paths need not be k-locally rainbow graphs, hence they are nearly k-locally rainbow graphs.*

Observation 4.4. C_4 *is not a locally rainbow graph, and it is not at all a nearly locally rainbow graph.*

Observation 4.5. *Any complete graph* K_n *is neither locally rainbow nor nearly locally rainbow graphs.*

Observation 4.6. *Let G be a k-locally rainbow graph on n vertices and let* $lc_k(G) = r$. *Then* $V(G) = V_1 \cup V_2 \cup \cdots \cup V_r$ *where* V_i *is the set of all vertices coloured by i where* $1 \leq i \leq r$. *we have,*

1. *Since G is a k-locally rainbow graph each* V_i *is a nonempty set and they are independent in G.*
2. *It is obvious that* $r \leq n$.
3. $u \in V_i$ *can be adjacent to only one vertex* $v \in V_{i+1}$. *Otherwise, suppose the vertex* $u \in V_i$ *is adjacent to two vertices* $v, w \in V_{i+1}$, *then the set* $S = \{u, v, w\}$ *does not satisfy the k-local colour condition.*
4. $u_1 \in V_1, u_2 \in V_2, \cdots, u_r \in V_r$ *cannot form a cycle* C_r *as again they are not satisfying the k-local colour condition when* $k = r$.

Theorem 4.7. *Let G be a graph with local chromatic number* $3(\chi_l(G) = 3)$, *then G is a triangle free graph.*

Proof. Let G be a graph on n vertices with local chromatic number 3. Suppose G has a triangle. Let H be the subgraph which is a triangle formed by the vertices $u, v, w \in G$. The local chromatic number of the triangle H is 4. As H is a subgraph of G, $\chi_l(H) \leq \chi_l(G)$, but $\chi_l(H) = 4$ where as $\chi_l(G) = 3$. Hence G cannot have a triangle. Therefore, if $\chi_l(G) = 3$, then G is a triangle free graph. □

The converse of the above theorem is not true. We have the following example.

Example 4.8. *Grötzsch graph is a triangle free graph, but the local chromatic number of it is 4.[6] It is a locally rainbow graph.*

Suppose G is a k-locally rainbow colourable graph of order n, the following result gives a strategy to find an independent set of size n in $G \times K_r$.

Theorem 4.9. *Let G be a graph on n vertices and is k-locally rainbow colourable with $lc_k(G) = r$, then the cartesian product $G \times K_r$ has an independent set of size n.*

Proof. Let $lc_k(G) = r$ where G is k-locally rainbow colourable. Let $c_1 : V \to \{1, 2, 3, \cdots, r\}$ be the k-local coloring ,then $V(G) = V_1 \cup V_2 \cup \cdots \cup V_r$ where V_i is the set of all vertices coloured by i, $1 \leq i \leq r$. Consider the complete graph K_r, on r vertices $v_1, v_2, \cdots, v_r$, Let $c_2 : V(K_r) \to \{1, 2, \cdots, r\}$ be the colouring defined by, $c_2(v_i) = i$, $v_i \in V(K_r)$ for $1 \leq i \leq r$. Consider the caretisian product $G \times K_r$. Let S be a subset of $V(G \times K_r)$ and define S as $\{(u, v) \in V(G \times K_r)/c_1(u) = c_2(v)\}$. As each vertex of G has a colour from the set $\{1, 2, 3, \cdots, r\}$,and S has n pair of vertices (u, v) we have, $|S| = n$.
Claim: S is an independent set. Suppose $(u_1, v_1), (u_2, v_2) \in S$ and there is an edge $(u_1, v_1)(u_2, v_2)$ in $V(G \times K_r)$ then by the definition of the cartesian product, there are two possibilities.
(1) $u_1 = u_2$ and $v_1 v_2$ is an edge in K_r. But $u_1 = u_2$ implies $c_1(u_1) = c_1(u_2)$. As $(u_1, v_1), (u_2, v_2) \in S$ and by the definition of S, $c_1(u_1) = c_2(v_1) = c_1(u_2) = c_2(v_2)$, which is a contradiction, as $c_2(v_1)$ and $c_2(v_2)$ are not equal in K_r.
(2) $u_1 u_2$ is an edge in G and $v_1 = v_2$. As $u_1 u_2$ is an edge in G they will have different colours (say) $i, j \in \{1, 2, \cdots, r\}$. respectively. As $(u_1, v_1), (u_2, v_2) \in S$ and by the definition of S, $c_2(v_1) = c_1(u_1) = i, c_1(u_2) = j = c_2(v_2)$, which is a contradiction, as $v_1 = v_2$. Therefore, there cannot be an edge between any two vertices in S and hence S is an independent set. □

5. Conclusion

In this article the k-local chromatic number of the Jahangir graph $J_{n,m}$ is determined. Some characteristics of k-locally rainbow graphs and rainbow graphs are studied.

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