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Some Results on Rosa-type Labelings of Graphs

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Abstract

Labelings that are used in graph decompositions are called *Rosa-type* labelings. The γ-labeling of an almost-bipartite graph is a natural generalization of an α-labeling of a bipartite graph. It is known that if a bipartite graph G with m edges possesses an α-labeling or an almost-bipartite graph G with m edges possesses a γ-labeling, then the complete graph K_{2mx+1} admits a cyclic G-decomposition. A variation of an α-labeling is introduced in this paper by allowing additional vertex labels and some conditions on edge labels and show that whenever a bipartite graph G admits such a labeling, then there exists a supergraph H of G such that H is almost-bipartite and H has a γ-labeling.

Keywords: Graph Labeling, Rosa-type labeling, α-labeling, γ-labeling, ρ-labeling

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1. Introduction

Let G be a simple connected bipartite graph with bipartition (A, B) and m edges. Let $V(G)$ and $E(G)$ denote the vertex set of G and the edge set of G, respectively. Let $f : V(G) \rightarrow \{0, 1, 2, \dots\}$ be a one-to-one function. Let $f(V(G)) = \{f(v) : v \in V(G)\}$. Define a function $\bar{f} : E(G) \rightarrow \{1, 2, 3, \dots\}$ by $\bar{f}(e) = |f(u) - f(v)|$, where $e = uv \in E(G)$. We call f(v) and $\bar{f}(e)$ *labels* of the vertex v and edge e respectively. Let $\bar{E}(G) = \{\bar{f}(e) : e \in E(G)\}$.

If a and b are integers with $a < b$, we denote $\{a, a+1, a+2, \dots, b\}$ by $[a, b]$.

Rosa introduced a hierarchy of labelings including α-labeling of a bipartite graph.[3] Blinco *et al.* introduced the notation of γ-labeling

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of an almost-bipartite graph as a natural generalization of an α-labeling of a bipartite graph.[1]

Joseph Gallian gives a comprehensive and updated dynamic survey on general graph labeling.[2]

Consider the following conditions.

$\ell_1 : f(V(G)) \subseteq [0, 2m]$,

$\ell_2 : f(V(G)) \subseteq [0, m]$,

$\ell_3 : \bar{E}(G) = \{x_1, x_2, \ldots, x_m\}$, where for each $i \in [1, m]$ either $x_i = i$ or $x_i = 2m + 1 - i$,

$\ell_4 : \bar{E}(G) = [1, m]$.

ℓ_5 : there exists an integer λ such that $f(a) \leq \lambda$ for all $a \in A$ and $f(b) > \lambda$ for all $b \in B$.

A labeling satisfying the conditions ℓ_1 and ℓ_3 is called a *ρ-labeling*.
A labeling that satisfies ℓ_2, ℓ_4 and ℓ_5 is called an *α-labeling*.
A non-bipartite graph G is said to be *almost-bipartite* if $G - e$ is bipartite for some $e \in E(G)$. If G is almost-bipartite with $e = \hat{b}c$, then G is tripartite and $V(G)$ can be partitioned into three sets A, B and $\{c\}$ such that $\hat{b} \in B$ and e is the only edge joining an element of B to c.
Let G be an almost-bipartite graph with m edges with vertex tripartition A, B, $\{c\}$ as above. A labeling h of the vertices of G is called a *γ-labeling* of G if the following conditions hold.
g_1: The function h is a ρ-labeling of G.
g_2: If av is an edge of G with $a \in A$, then $h(a) < h(v)$.
g_3: $h(c) - h(\hat{b}) = m$.

It is known that

(i) Let G be a graph with m edges. There exists a cyclic G-decomposition of K_{2m+1} if and only if G has a ρ-labeling.[3]

(ii) Let G be a bipartite graph with m edges that has an α-labeling. Then, for all positive integers x, G divides K_{2mx+1}.[3]

(iii) Let G be an almost-bipartite graph with m edges having a γ-labeling. Then, for all positive integers x, G divides K_{2mx+1}.[1]

2. Results

If $\bar{f}$ is one-to-one and there exists an integer λ such that $f(a) \leq \lambda$ for all $a \in A$ and $f(b) > \lambda$ for all $b \in B$, then we say that f is an *$\alpha(\bar{E}(G))$-labeling* of G. When $f(V(G)) = [0, m]$ an $\bar{E}(G) = [1, m]$, and $\alpha(\bar{E}(G))$-labeling of G is the usual α-labeling of G.

We now prove some results related to α-labeling and λ-labeling.

Theorem 2.1. *Let f be an $\alpha(\bar{E}(G))$-labeling of G with*

$$f(V(G)) \subseteq [0, m + 1]$$

and

$$\bar{E}(G) = [1, m+1] \setminus \{d\},$$

where $d \in [2, m]$. *Suppose*

$$d = m + 2 - \lambda.$$

Then there exist vertices $a \in A$ *and* $b \in B$, *such that the almost-bipartite graph* G^* *obtained from G by defining*

$$V(G^*) = V(G) \cup \{v\}$$

and

$$E(G^*) = E(G) \cup \{av, vb\}$$

has a γ*-labeling.*

Proof. As $d \neq 1$ and $d \neq m+1$, there exist edges $e' = x'y'$ with $x' \in A$, $y' \in B$, $f(x') = \lambda$, $f(y') = \lambda+1$, $\bar{f}(e') = 1$ and $e'' = x''y''$ with $x'' \in A$, $y'' \in B$, $f(x'') = 0$, $f(y'') = m+1$ and $\bar{f}(e'') = m+1$. Take $a = x''$ and $b = y'$. Then, $f(a) = 0$ and $f(b) = \lambda + 1$.

Define

$$g : V(G^*) \to [0, 2m+4]$$

by

$g(w) = f(w)$ for every $w \in V(G)$

and

$g(v) = \lambda + m + 3.$

Then

$$\bar{g}(av) = (2m+5) - (\lambda + m + 3) = m + 2 - \lambda = d$$

and

$$\bar{g}(vb) = (\lambda + m + 3) - (\lambda + 1) = m + 2.$$

Hence,

$$\bar{E}(G^*) = \bar{E}(G) \cup \{d, m+2\} = [1, m+2].$$

Clearly, g is one-to-one and hence g is a γ-labeling of G^*.

□

Theorem 2.2. *Let f be an* $\alpha(\bar{E}(G))$*-labeling of G with*

$$f(V(G)) \subseteq [0, m+1]$$

and

$$\bar{E}(G) = [1, m+1] \setminus \{d\}.$$

Suppose

$$d = m + 2 - \lambda$$

and there exists an i in $[0, \lambda]$ such that both i and $\lambda+1+i$ are in $f(V(G))$. Then there exist vertices $a \in A$ and $b \in B$, such that the almost-bipartite graph G^* obtained from G by defining

$$V(G^*) = V(G) \cup \{v\}$$

and

$$E(G^*) = E(G) \cup \{av, vb\}$$

has a γ-labeling.

Proof. Take a and b so that $f(a) = i$ and $f(b) = \lambda + 1 + i$.

Define

$$g : V(G^*) \to [0, 2m + 4]$$

by

$g(w) = f(w)$ for every $w \in V(G)$

and

$g(v) = \lambda + m + 3 + i$.

Then

$$\bar{g}(av) = (2m + 5 + i) - (\lambda + m + 3 + i) = m + 2 - \lambda = d$$

and

$$\bar{g}(vb) = (\lambda + m + 3 + i) - (\lambda + 1 + i) = m + 2.$$

Hence,

$$\bar{E}(G^*) = \bar{E}(G) \cup \{d, m + 2\} = [1, m + 2].$$

Clearly, g is one-to-one and hence g is a γ-labeling of G^*. □

Remark 2.3. *Theorem 2.2 is a generalization of Theorem 2.1. To see this, take $i = 0$ in Theorem 2.2.*

Theorem 2.4. *Let f be an $\alpha(\bar{E}(G))$-labeling of G with*

$$f(V(G)) \subseteq [0, m + 3] \setminus \{m + 2\}$$

and

$$\bar{E}(G) = [1, m + 2] \setminus \{d, m + 4 - d\}$$

for some $d \in [2, m + 2]$. Then, the almost-bipartite graph $G + K_3$, the disjoint union of G and K_3, has a γ-labeling.

Proof. Let

$$V(G + K_3) = V(G) \cup \{u, v, w\}$$

and

$$E(G + K_3) = E(G) \cup \{uv, vw, wu\}.$$

Define

$$g : V(G + K_3) \to [0, 2m + 6]$$

by

$g(x) = f(x) + d + 1$ for every $x \in V(G)$,
$g(u) = 0$,
$g(v) = d$,

and

$g(w) = m + d + 3$.

Then,

$$\bar{g}(uv) = d,$$

$$\bar{g}(vw) = (m + d + 3) - d = m + 3,$$

and

$$\bar{g}(wu) = (2m + 7) - (m + d + 3) = m + 4 - d.$$

Hence,

$$\bar{E}(G + K_3) = \bar{E}(G) \cup \{d, m + 4 - d, m + 3\} = [1, m + 3].$$

Clearly, g is one-to-one and hence g is a γ-labeling of $G + K_3$. □

Theorem 2.5. *Let f be an $\alpha(\bar{E}(G))$-labeling of G with*

$$f(V(G)) \subseteq [0, m + 3] \setminus \{m + 1\}$$

and

$$\bar{E}(G) = [1, m + 2] \setminus \{d, m + 4 - d\}$$

for some $d \in [2, m + 1]$. Then, the almost-bipartite graph $G + K_3$, the disjoint union of G and K_3, has a γ-labeling.

Proof. In the proof of previous theorem, take $g(x) = f(x) + d + 2$ for every $x \in V(G)$.

For integers $r \geq s \geq 2$, let $T_{r,s}$ be the double star with

$$V(T_{r,s}) = \{x, y, x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_s\}$$

and

$$E(T_{r,s}) = \{xy, xx_1, xx_2, \ldots, xx_r, yy_1, yy_2, \ldots, yy_s\}.$$

Then, $m = r + s + 1$.

We now see two illustrations associated with the above theorems.

Illustration 1 Define

$$f : V(T_{r,s}) \to [0, m + 1]$$

by

$f(x) = 0$,
$f(y_i) = i$ for every $i \in [1, s]$,
$f(y) = s + 1$,

and

$\{f(x_1), f(x_2), \ldots, f(x_r)\} = [s+2, r+s+2] \setminus \{r+3\}$.
So $\lambda = s$ and $d = m+2-\lambda = (r+s+1)+2-s = r+3$.

Then f satisfies the hypothesis of Theorem 2.1. Hence, by the proof of Theorem 2.1, the almost-bipartite graph $T^*_{r,s}$ obtained from $T_{r,s}$ by defining

$$V(T^*_{r,s}) = V(T_{r,s}) \cup \{v\}$$

and

$$E(T^*_{r,s}) = E(T_{r,s}) \cup \{xv, vy\}$$

has a γ-labeling. Consequently, for all positive integers x, $T^*_{r,s}$ divides the complete graph $K_{2(r+s+3)x+1}$.

We apply Theorem 2.2 with $i = 1$.

As $r \geq s$, $s+2 \neq r+3$, and hence $f(x_1) = s+2$. So, f satisfies the hypothesis of Theorem 2.2. By the proof of Theorem 2.2, the almost-bipartite graph $T^{**}_{r,s}$ obtained from $T_{r,s}$ by defining

$$V(T^{**}_{r,s}) = V(T_{r,s}) \cup \{v\}$$

and

$$E(T^{**}_{r,s}) = E(T_{r,s}) \cup \{y_1v, vx_1\}$$

has a γ-labeling. Consequently, for all positive integers x, $T^{**}_{r,s}$ divides the complete graph $K_{2(r+s+3)x+1}$.

Illustration 2 Define

$$f : V(T_{r,s}) \to [0, m+3]$$

by

$f(x) = 0$,
$\{f(y_1), f(y_2), \ldots, f(y_s)\}, = [1, s+1] \setminus \{s-1\}$,
$f(y) = s+2$,

and

$\{f(x_1), f(x_2), \ldots, f(x_r)\}, = [s+3, r+s+4] \setminus \{r+s+2\}$.
Then f satisfies the hypothesis of Theorem 2.3 with $d = 3$. Hence, by Theorem 2.3, the almost-bipartite graph $T_{r,s} + K_3$ has a γ-labeling. Consequently, for all positive integers x, $T_{r,s} + K_3$ divides the complete graph $K_{2(r+s+4)x+1}$. □

The results obtained may be helpful to enlarge the known class of γ-labelable graphs.

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