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Connected Weak Edge Detour Number of a Graph

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Abstract

Certain general properties of the *detour distance*, *weak edge detour set*, *connected weak edge detour set*, *connected weak edge detour number* and *connected weak edge detour basis* of graphs are studied in this paper. Their relationship with the detour diameter is discussed. It is proved that for each pair of integers k and n with $2 \leq k \leq n$, there exists a connected graph G of order n with $cdn_w(G) = k$. It is also proved that for any three positive integers R, D, k such that $k \geq D$ and $R < D \leq 2R$, there exists a connected graph G with $rad_D\ G = R$, $diam_D\ G = D$ and $cdn_w(G) = k$.

Keywords: Detour, Detour number, Weak edge detour number, Connected weak edge detour number

Mathematics Subject Classification (2010): 05C12

1. Introduction

Graphs are discrete structures that represent objects and their relations among them. For a *graph* $G = (V, E)$, with the vertex (object) set V and edge set, i.e., the set of relations, E, the order and size of G are denoted by n and m respectively. For basic definitions and terminologies we refer to [4, 1]. Throughout this paper G denotes a finite undirected connected simple graph with at least two vertices.

For vertices u and v in G, the *distance* $d(u, v)$ is the length of a shortest $u-v$ path in G. A $u-v$ path of length $d(u, v)$ is called a $u-v$ *geodesic*. For a vertex v of G, the *eccentricity* $e(v)$ is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices

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of G is the *radius*, *rad* G and the maximum eccentricity is its *diameter*, *diam* G of G.

The *detour distance* $D(u, v)$ is the length of a longest $u-v$ path in G for vertices u and v in G. A $u-v$ path of length $D(u, v)$ is called a $u-v$ *detour*. For a vertex v of G, the *detour eccentricity* $e_D(v)$ is the detour distance between v and a vertex farthest from v. The *detour radius*, rad_D G of G is the minimum detour eccentricity among the vertices of G, while the *detour diameter*, $diam_D$ G of G is the maximum detour eccentricity among the vertices of G. These concepts were studied by Chartrand *et al.* [2]

A vertex x is said to lie on a $u-v$ detour P if x is a vertex of P including the vertices u and v. A set $S \subseteq V$ is called a *detour set* if every vertex v in G lies on a detour joining a pair of vertices of S. The *detour number* $dn(G)$ of G is the minimum order of a detour sets and any detour set of order $dn(G)$ is called a *detour basis* of G. A vertex v that belongs to every detour basis of G is a *detour vertex* in G. If G has a unique detour basis S, then every vertex in S is a detour vertex in G.[3]

A set $S \subseteq V$ is called a *weak edge detour set* of G if every edge in G has both its ends in S or it lies on a detour joining a pair of vertices of S. The *weak edge detour number* $dn_w(G)$ of G is the minimum order of its weak edge detour sets and any weak edge set of order $dn_w(G)$ is called a *weak edge detour basis* of G. These concepts were studied by Santhakumaran and Athisayanathan.[5]

A set $S \subseteq V$ is called a *connected detour set* of G if S is a detour set of G and the subgraph $G\langle S\rangle$ induced by S is connected. The *connected detour number* $cdn(G)$ of G is the minimum order of its connected detour sets and any connected detour set of order $cdn(G)$ is called *connected detour basis* of G.[6] This motivated us to introduce and investigate the concepts of *connected weak edge detour set* and *connected weak edge detour number* of a graph G.

The following theorems are used in this paper for proving the results.

Theorem 1.1. *[3] Every end-vertex of a non-trivial connected graph G belongs to every detour set of G. Also if the set S of all end-vertices of G is a detour set, then S is the unique detour basis for G.*

Theorem 1.2. *[5] Every end-vertex of a non-trivial connected graph G belongs to every weak edge detour set of G. Also if the set S of all end-vertices of G is a weak edge detour set, then S is the unique weak edge detour basis for G.*

Theorem 1.3. *[5] If T is a non-trivial tree with k end-vertices, then $dn(T) = dn_w(T) = k$.*

2. Connected Weak Edge Detour Number of a Graph

Definition 2.1. *Let $G = (V, E)$ be a connected graph with at least two vertices. A set $S \subseteq V$ is a connected weak edge detour set of G if S is a weak edge detour set of G and the subgraph $\langle S \rangle$ induced by S is connected. The connected weak edge detour number $cdn_w(G)$ of G is the minimum order of its connected weak edge detour sets and any connected weak edge detour set of order $cdn_w(G)$ is called a connected weak edge detour basis of G.*

Example 2.2. *For the graph G given in Figure 2.1, it is clear that no two element subset of V is a connected weak edge detour set of G.The set $S = \{v_1, v_2, v_3\}$ is a connected weak edge detour basis of G so that $cdn_w(G) = 3$. The set $S_1 = \{v_1, v_2, v_4\}$ and $S_2 = \{v_1, v_3, v_5\}$ are also connected weak edge detour bases of G. Thus there can be more than one connected weak edge detour basis for a graph G.*

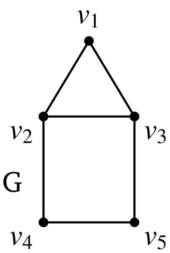


Figure 2.1: G

Remark 2.3. *Every connected weak edge detour set is a weak edge detour set but the converse is not true. For the graph G given in figure 2.1, the set $U = \{v_1, v_4, v_5\}$ is a weak edge detour set but not a connected weak edge detour set of G.*

Example 2.4. *For the graph G given in Figure 2.2, the set $S_1 = \{v_2, v_3\}$ is a connected weak edge detour basis for G so that $cdn_w(G) = dn_w(G) = 2$.*

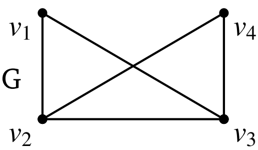


Figure 2.2: G

Theorem 2.5. *For any graph G of order $n \geq 2$, $2 \leq cdn_w(G) \leq n$.*

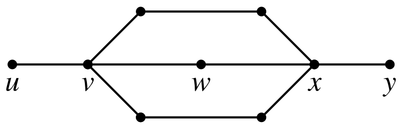
Proof. A connected weak edge detour set needs at least two vertices so that $cdn_w(G) \geq 2$ and the set of all vertices of G is a connected weak edge detour set of G so that $cdn_w(G) \leq n$. Thus $2 \leq cdn_w(G) \leq n$. □

Remark 2.6. *The bounds in Theorem 2.5 are sharp. For the complete graph K_2, $cdn_w(K_2) = 2$. The set of all vertices of path P_n $(n \geq 2)$ is* *its unique connected weak edge detour set so that $cdn_w(G) = n$. Also the inequalities in Theorem 2.5 can be strict. For the graph G given in Figure 2.1, $n = 5$, $cdn_w(G) = 3$ so that $2 < cdn_w(G) < n$. Thus the complete graph K_2 has the smallest possible connected weak edge detour number 2 and the non-trivial paths have the largest possible connected weak edge detour number n.*

Definition 2.7. *A vertex v in a graph G is a connected weak edge detour vertex if v belongs to every connected weak edge detour basis of G. If G has a unique connected weak edge detour basis S, then every vertex in S is a connected weak edge detour vertex of G.*

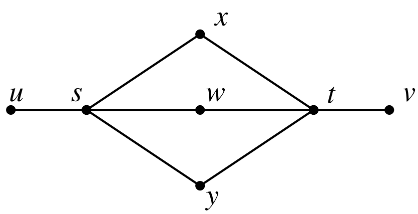
Example 2.8. *For the graph G given in Figure 2.3, $S = \{u, v, w, x, y\}$ is the unique connected weak edge detour basis so that every vertex of S is a connected weak edge detour vertex of G.*

Figure 2.3: G



Example 2.9. *For the graph G given Figure 2.4, $S_1 = \{u, s, w, t, v\}$, $S_2 = \{u, s, x, t, v\}$ and $S_3 = \{u, s, y, t, v\}$ are the connected weak edge detour bases of G so that u, s, t and v are the connected weak edge detour vertices of G.*

Figure 2.4: G



In the following theorems we show that there are certain vertices in a non-trival connected graph G that are connected weak edge detour vertices of G.

Theorem 2.10. *Every end-vertex of a non-trivial connected graph G belongs to every connected weak edge detour set of G.*

Proof. Let v be an end-vertex of G and uv an edge in G incident with v. Then uv is either an initial edge or the terminal edge of any detour containing the edge uv. Hence it follows that v belongs to every connected weak edge detour set of G. □

Theorem 2.11. *Let G be a connected graph with cut-vertices and S a connected weak edge detour set of G. Then for any cut-vertex v of G, every component of $G - v$ contains an element of S.*

Proof. Let v be a cut-vertex of G such that one of the components, say C of $G - v$ contains no vertex of S. Then by Theorem 2.10, C does not contain any end-vertex of G. Hence C contains at least one edge, say uw. Since S is a connected weak edge detour set there exists vertices $x, y \in S$ such that uw lies on some $x - y$ detour $P : x = u_0, u_1, \ldots, u, w, \ldots u_t = y$ in G or both the ends u and w of the edge uw are in S. Suppose that uw lies on the detour P. Let P_1 be the $x - u$ subpath of P and P_2 be the $u - y$ subpath of P. Since v is a cut-vertex of G both P_1 and P_2 contain v so that P is not a detour, which is a contradiction. Suppose that u and w are in S, then C contains vertices of S, which is again a contradiction. □

Theorem 2.12. *Let G be a connected graph with cut-vertices. Then every cut-vertex of G belongs to every connected weak edge detour set of G.*

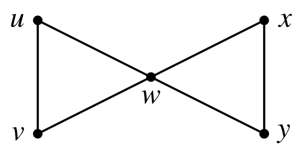
Proof. Let G be a connected graph and v be a cut-vertex of G. Let G_1, $G_2, \ldots, G_k$ $(k \geq 2)$ be the components of $G-v$. Let S be any connected weak edge detour set of G. Then by Theorem 2.11, S contains at least one element from each component G_i $(1 \leq i \leq k)$ of $G - v$. Since $\langle S \rangle$ is connected it follows that $v \in S$. □

Corollary 2.13. *All the end-vertices and the cut-vertices of a connected graph G belong to every connected weak edge detour set of G.*

Proof. Proof is immediate from the Theorems 2.10 and 2.12. □

Remark 2.14. *For the graph G given in Figure 2.5, $S_1 = \{u, w, x\}$, $S_2 = \{u, w, y\}$, $S_3 = \{v, w, x\}$ and $S_4 = \{v, w, y\}$ are the four connected weak edge detour bases. The cut vertex w belongs to every connected weak edge detour basis so that the cut-vertex w is the unique connected weak edge detour vertex of G.*

Figure 2.5: G



Corollary 2.15. *If T is a tree of order $n \geq 2$, then $cdn_w(T) = n$.*

Proof. Corollary 2.13 gives the proof. □

Corollary 2.16. *For any connected graph G with k end-vertices and l cut-vertices, $max\{2, k + l\} \leq cdn_w(G) \leq n$.*

Proof. The Theorem 2.5 and the corollary 2.13 give the proof. □

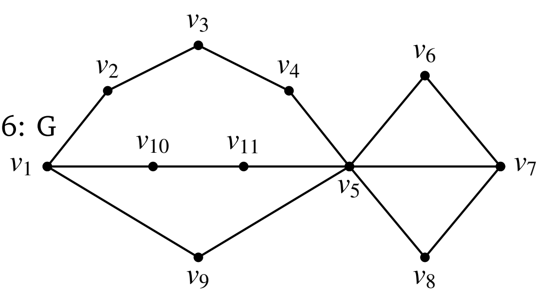
For the graph H and an integer $k \geq 1$, we write kH for the union of the k disjoint copies of H.

Theorem 2.17. *Let $G = (K_{n_1} \cup K_{n_2} \cup \ldots \cup K_{n_r} \cup kK_1) + v$ be a block graph of order $n \geq 4$ such that $r \geq 1$, each $n_i \geq 2$ and $n_1 + n_2 + \ldots + n_r + k = n - 1$. Then $cdn_w(G) = r + k + 1$.*

Proof. Let $u_1, u_2, \ldots u_k$ be the end-vertices of G. Let S be any connected weak edge detour set of G. Then by Corollary 2.13, $v \in S$ and $u_i \in S (1 \leq i \leq k)$. Also by Theorem 2.11, S contains a vertex from each component K_{n_i} $(1 \leq i \leq r)$. Now choose exactly one vertex v_i from each K_{n_i} such that $v_i \in S$. Then $|S| \geq r + k + 1$. Let $T = \{v, v_1, v_2, \ldots v_r, u_1, u_2, \ldots, u_k\}$. Since every edge in G has both its ends in T or it lies on a detour joining a pair of vertices of T, it follows that T is a weak edge detour basis of G. Also, since $\langle T \rangle$ is connected, $cdn_w(G) = r + k + 1$. □

Remark 2.18. *If the blocks of the graph G in Theorem 2.17 are not complete, then the theorem is not true. For the graph G given in Figure 2.6 there are two blocks and $\{v_4, v_9, v_5, v_7\}$ is a connected weak edge detour basis so that $cdn_w(G) = 4$.*

Figure 2.6: G



Theorem 2.19. *Let G be the complete graph K_n $(n \geq 2)$. Then a set $S \subseteq V$ is a connected weak edge detour basis of G if and only if S consists of any two vertices of G.*

Proof. Let G be the complete graph K_n $(n \geq 2)$ and $S = \{u, v\}$ be any set of two vertices of G. It is clear that $D(u, v) = n - 1$. Let $xy \in E$. If $xy = uv$, then both its ends are in S. Let $xy \neq uv$. If $x \neq u$ and $y \neq v$, then the edge xy lies on the $u - v$ detour $P : u, x, y, \ldots, v$ of length $n - 1$. If $x = u$ and $y \neq v$, then the edge xy lies on the $u - v$ detour $P : u = x, y, \ldots, v$ of length $n - 1$. Hence S is a connected weak edge detour of G. Since $|S| = 2$, S is a connected weak edge detour basis of G.

Conversely, let S be a connected weak edge detour basis of G. Let S' be any set consisting of two vertices of G. Then as in the first part of this theorem S' is a connected weak edge detour basis of G. Hence $|S| = |S'| = 2$ and it follows that S consists of any two vertices of G. □

Theorem 2.20. *Let G be a cycle of order $n \geq 3$. Then a set $S \subseteq V$ is a connected weak edge detour basis of G if and only if S consists of any two adjacent vertices of G.*

Proof. Let $S = \{u, v\}$ be any set of two adjacent vertices of G. It is clear that $D(u, v) = n-1$. Then every edge $e \neq uv$ of G lies on the $u-v$ detour and the both ends of the edge uv belong to S so that S is a connected weak edge detour set of G. Since $|S| = 2$, S is a connected weak edge detour basis of G.

Conversely, assmume that S is a connected weak edge detour basis of G. Let S' be any set of two adjacent vertices of G. Then as in the first part of this theorem S' is a connected weak edge detour basis of G. Hence $|S| = |S'| = 2$. Let $S = \{u, v\} \subseteq V$. If u and v are not adjacent, it is clear that u and v are not connected. Thus S consists of any two adjacent vertices of G. □

Theorem 2.21. *Let G be the complete bipartite graph $K_{m,n}$ $(2 \leq m \leq n)$. Then a set $S \subseteq V$ is a connected weak edge detour basis of G if and only if S consists of any two adjacent vertices of G.*

Proof. Let X and Y be the bipartite sets of G with $|X| = m$ and $|Y| = n$. Let $S = \{u, v\}$, where $u \in X$ and $v \in Y$ be any two adjacent vertices of G. It is clear that $D(u, v) = 2m - 1$. Then every edge $e \neq uv$ of G lies on the uv-detour and the both ends of the edge uv belongs to S so that S is a connected weak edge detour set of G. Since $|S| = 2$, S is a connected weak edge detour basis of G.

Conversely, assume that S is a connected weak edge detour basis of G. Let S' be any set of two adjacent vertices of G. Then as in the first part of this theorem S' is a connected weak edge detour basis of G. Hence $|S| = |S'| = 2$. Let $S = \{u, v\} \subseteq V$. If u and $v \in X$ or Y it is clear that u and v are not connected. Thus S consists of any two adjacent vertices of G. □

Corollary 2.22. *(a) If G is the complete graph K_n, then $cdn_w(G) = 2$.*

(b) If G is the complete bipartite graph $K_{m,n}$ $(2 \leq m \leq n)$, then $cdn_w(G) = 2$.

(c) If G is the cycle C_n, then $cdn_w(G) = 2$.

Proof. (a) It follows from Theorem 2.19.

(b) It follows from Theorem 2.21.

(c) It follows from Theorem 2.10.

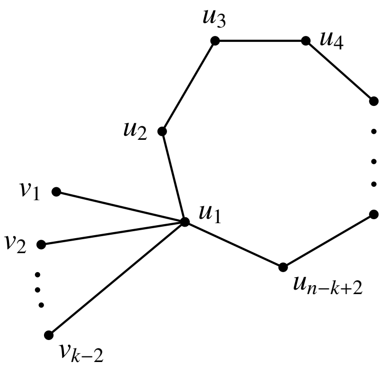
□

The following theorems give realization results.

Theorem 2.23. *For each pair of integer k and n with $2 \leq k \leq n$, there exists a connected graph G of order n with $cdn_w(G) = k$.*

Proof. **Case 1.** $k = n$. Then any tree of order n has the desired property by Corollary 2.15.
Case 2. $2 = k < n$, the cycle C_n has the desired property by Corollary 2.22 (c).
Case 3. $2 < k < n$. Let G be the graph obtained from the cycle $C_{n-k+2} : u_1, u_2, ...u_{n-k+2}, u_1$ of order $n - k + 2$ by adding $k - 2$ new vertices $v_1, v_2, \ldots v_{k-2}$ and joining each vertex v_i $(1 \leq i \leq k - 2)$ to u_1. The resulting graph G is connected of order n and is shown in Figure 2.7. Now we show that $cdn_w(G) = k$. Let $S = \{u_1, v_1, v_2, ...v_{k-2}\}$ be the set of all end-vertices together with the cut-vertex u_1 of G. It is clear that S is not a connected weak edge detour set of G. Let $T = S \cup \{u_2\}$. Then every edge of G has both its ends in T or it lies on a detour joining a pair of vertices of T and also <T> is a connected so that T is a connected weak edge detour basis of G, so that $cdn_w(G) = k$. □

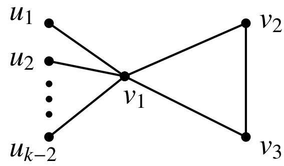
Figure 2.7: G



Theorem 2.24. *For each positive integer $k \geq 2$ there exists a connected graph G and a vertex v of degree k in G such that v belongs to a connected weak edge detour basis of G and $cdn_w(G) = k$.*

Proof. **Case 1.** $k = 2$, the complete graph K_3 has a desired properties by Corollary 2.22 (a).
Case 2. $k > 2$, let G be the graph obtained from the complete graph K_3, where $V(K_3) = \{v_1, v_2, v_3\}$ by adding $k-2$ new vertices $u_1, u_2, ..., u_{k-2}$ and joining $u_i(1 \leq i \leq k - 2)$ to v_1. The resulting graph G is connected of order n and is shown in the Figure 2.8. Then $deg_G v_1 = k$. Let $S = \{u_1, u_2, ...u_{k-2}, v_1\}$ be the set of all end-verties and cut-verties. However, by Corollary 2.13, S is not a connected weak edge detour set of G. Let $T = S \cup \{v\}$, where $v \in \{v_2, v_3\}$ is a vertex in K_3. Then T is a connected weak edge detour basis of G and hence so that $cdn_w(G) = k$. □

Figure 2.8: G



Theorem 2.25. *For every pair of positive integer a, b with $2 \leq a \leq b$, there exists a connected graph G such that $dn_w(G) = a$ and $cdn_w(G) = b$.*

Proof. **Case 1:** $a = b$, we have the following two sub cases.
Sub case (i): $a = 2$, the complete graph K_2 has the desired property.
Sub case (ii): $a > 2$. Let $C_3 : u_1, u_2, u_3$ be the cycle of length 3. Now, by adding $a - 2$ new vertices $v_1, v_2, ...v_{a-2}$ and joining the vertex u_2 as shown in the Figure 2.9. Let $S = \{v_1, v_2, ...v_{a-2}, u_2\}$ be the set of all end vertices and cut-verties of G. It is clear that S is not a weak edge detour set of G. Let $T = S \cup \{u\}$, where $u \in \{u_1, u_3\}$ is a vertex in C_3. Then T is a weak edge detour basis of G so that $dn_w(G) = a$. Also the sub graph $\langle T \rangle$ induced by T is connected so that $cdn_w(G) = a$.

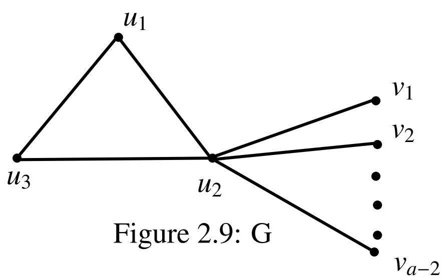


Figure 2.9: G

Case 2: $a < b$. Let G be any tree with a end -vertices and $b - a$ cut-vertices. Then by Theorem 1.3, $dn_w(G) = a$ and by Corollary 2.15, $cdn_w(G) = b$. □

3. Connected Weak Edge Detour Number and Detour Diameter of a graph

In [3], an upper bound for the detour number, of a graph is given in terms of its order and detour diameter D as follows:
Proposition A[3] If G is a non-trival connected graph of order $n \geq 3$ and detour diameter D, then $dn(G) \leq n - D + 1$.

Remark 3.1. *In the case of weak edge detour number $dn_w(G)$ of a graph G it is show in* [5] *that, there are graphs G for which $dn_w(G) = n - D + 1$,*

$dn_w(G) > n - D + 1$ and $dn_w(G) < n - D + 1$. Similarly, in the case of connected weak edge detour number $cdn_w(G)$ of the graph G, we show that there are graphs for which $cdn_w(G) = n - D + 1$, $cdn_w(G) < n - D + 1$ and $cdn_w(G) > n - D + 1$. For the graph G given in Figure 3.1(a), $n = 6$, $D = 4$, $cdn_w(G) = 5$ so that $cdn_w(G) > n - D + 1$. For the graph G given in Figure 3.1(b), $n = 8$, $D = 4$ and $cdn_w(G) = 5$ so that $cdn_w(G) = n - D + 1$. For the graph G given in Figure 3.1(c), $n = 6$, $D = 4$ and $cdn_w(G) = 2$ so that $cdn_w(G) < n - D + 1$.

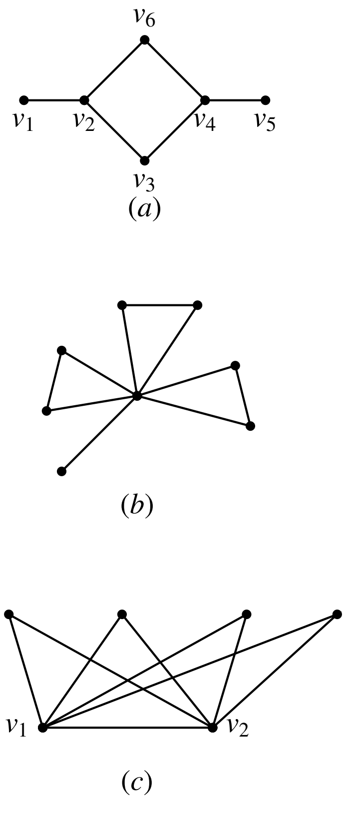


Figure 3.1: G

Theorem 3.2. *Let G be a connected graph of order $n \geq 2$. If $D = n - 1$, then $cdn_w(G) \geq n - D + 1$.*

Proof. For any graph G, $cdn_w(G) \geq 2$. Since $D = n - 1$, we have $n - D + 1 = 2$ and so $cdn_w(G) \geq n - D + 1$. □

Remark 3.3. *The converse of the Theorem 3.2 is not true. For the graph G given in Figure 3.1 (b), as in the Remark 3.1, $cdn_w(G) = n - D + 1$, but $D \neq n - 1$. Also for the graph G given in Figure 3.1 (a), as in the Remark 3.1, $cdn_w(G) > n - D + 1$, but $D \neq n - 1$.*

Theorem 3.4. *Let R, D, k be three positive integers such that $k > D$ and $R < D \leq 2R$. Then there exists a connected graph G such that $rad_D\ G = R$, $diam_D\ G = D$ and $cdn_w(G) = k$.*

Proof. **Case 1:** When $R = 1$ and $D = 2$, let $G = K_{1,k-1}$. Clearly $rad_D G = 1$, $diam_D G = 2$ and by corollary 2.15, $cdn(G) = k$.
Case 2: When $R \geq 2$ and $R < D \leq 2R$, we construct a graph G with the desired properties as follows: Let $C_{R+1} : v_0, v_1, \ldots v_R, v_0$ be a cycle of order $R + 1$ and let $P_{D-R+1} : u_0, u_1, \ldots, u_{D-R}$ be a path of order $D - R + 1$. Let H be the graph obtained from C_{R+1} and P_{D-R+1} by identifying v_0 of C_{R+1} with u_0 of P_{D-R+1}. The required graph G is obtained from H by adding $k - D + R - 2$ new vertices $w_1, w_2, \ldots, w_{k-D+R-2}$ to H and joining each $w_i (1 \leq i \leq k - D + R - 2)$ to the vertex u_{D-R-1} and is shown in Figure 3.2. Clearly, G is connected such that $rad_D G = R$ and $diam_D G = D$. Now, we show that $cdn_w(G) = k$. Let $S = \{u_0, u_1, \ldots u_{D-R-1}, u_{D-R}, w_1, w_2, \ldots w_{k-D+R-2}\}$ be the set of all cut-vertices and end-vertices. However, by Corollary 2.13, S is not a connected weak edge detour set of G. Let $T = S \cup \{v\}$, where $v \in \{v_R, v_1\}$ is a vertex in C_{R+1}. Then T is a connected weak edge detour basis of G so that $cdn_w(G) = k$.

□

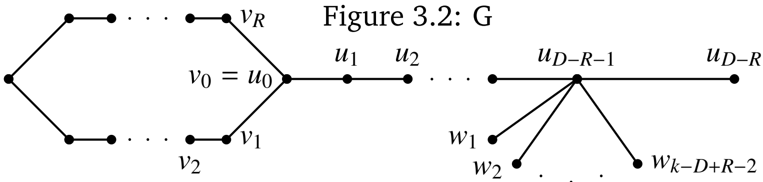


Figure 3.2: G

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