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Application of Ruscheweyh Derivative Operator on Meromorphic Functions in the Unit Disk

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Abstract

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In this paper we introduce a certain convolution operator $D^{\alpha+p-1}$ *on meromorphic functions of the form*

$$f(z)=\frac{1}{z^p}+\sum a_n z^n \text{ in } 0<|z|<1$$

and deal with the application of the above operator to certain differential inequalities involving $D^{\alpha+p-1}f$.

Introduction

In a recent paper[1] Lin Jinlin has introduced a differential operator defined by Convolution on functions of the form

$$f(z)=z^p+a_{p+1}z^{p+1}+\ldots \text{ (p a positive integer)}$$

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analytic in $|z|<1$ and, studied and obtained differential inequalities. In this paper we introduce a similar operator on functions $f \in M(p)$ where $M(p)$, p a positive integer denotes the class of meromorphic functions f of the form

$$f(z)=\frac{1}{z^p}+\frac{b_1}{z^{p-1}}+\ldots+\frac{b_{p-1}}{z}+\sum_0^{\infty} a_n z^n$$

which are analytic in the punctured disc $E_0=\{z:0<|z|<1\}$. We denote $f*g$ the convolution (or Hadamard product of $f,\ g \in M(p)$) defined as follows. Let

$$f(z)=\frac{1}{z^p}+\frac{b_1}{z^{p-1}}+\frac{b_2}{z^{p-2}}\ldots+\frac{b_{p-1}}{z}+\sum_0^{\infty} a_n z^n$$

$$g(z)=\frac{1}{z^p}+\frac{c_1}{z^{p-1}}+\frac{c_2}{z^{p-2}}\ldots+\frac{c_{p-1}}{z}+\sum_0^{\infty} d_n z^n$$

then

$$(f*g)(z)=\frac{1}{z^p}+\sum_{n=1}^{p-1}\frac{b_n c_n}{z^{p-n}}+\sum_{n=0}^{\infty} a_n d_n z^n$$

Using the above convolution, we define the operator $D^{\alpha+p-1}$ on f by

$$D^{\alpha+p-1}f(z)=\frac{1}{z^p(1-z)^{\alpha+p}}*f(z)$$

where $f(z)\in M(p)$ and α is any real number $>-p$.

For $\alpha=-(p-1)$, we get

$$D^0 f(z)=\frac{1}{z^p(1-z)}*f(z)=f(z)$$

so that D^0 acts as the identity operator on f.

We can easily prove that

$$D^{\alpha+p-1}f(z)=\frac{1}{z^p}\frac{(z^{\alpha+2p-1}f(z))^{(\alpha+p-1)}}{(\alpha+p-1)!}$$

If α is an integer not less than $-p+1$.
Therefore

$$D^{\alpha+p-1}f(z) \qquad =\frac{1}{z^p(1-z)^{\alpha+p}}*f(z)$$

$$=\frac{1}{z^p}\frac{(z^{\alpha+2p-1}f(z))^{(\alpha+p-1)}}{(\alpha+p-1)!}$$

We propose to study the application of this Convolution operator to certain differential inequalities.

Now we need the following.

Definition [1]

Let $H(\alpha)$ be the set of complex valued functions; *h(r,s,t)*; *h(r,s,t)*; $C^3 \to C$ (C is the complex plane) such that

i) $h(r,s,t)$ is continuous in a domain $D\subset C^3$;

ii) $(1,1,1)\in D$ and $|h(1,1,1)|<J(J>1)$;

iii)

$$\left|h\left(Je^{i\theta},\frac{(1+m)+(\alpha+p)Je^{i\theta}}{\alpha+p+1},\frac{1}{\alpha+p+2}\times\right.\right.$$

$$\left.\left.\cdot\left[2+m+(\alpha+p)Je^{i\theta}+\frac{m-m^2+(\alpha+p)mJe^{i\theta}+L}{1+m+(\alpha+p)Je^{i\theta}}\right]\right)\right|>J$$

whenever

$$\left(Je^{i\theta}, \frac{(1+m)+(\alpha+p)Je^{i\theta}}{\alpha+p+1}, \frac{1}{\alpha+p+2}\times\right.$$

$$\left.\left[2+m+(\alpha+p)Je^{i\theta}+\frac{m-m^2+(\alpha+p)mJe^{i\theta}+L}{1+m+(\alpha+p)Je^{i\theta}}\right]\right)\in D$$

with $Re \leq m\ (m-1)$ for real θ and for real $m \geq \frac{J-1}{J+1}$.

Main Result

First we state the following lemma due to Miller and Mocanu[2], which we use in the sequel.

Lemma

Let $W(z) = a + W_k z^k + \ldots$ be regular in the unit disc E with $W(z) \equiv a$ and $k \geq 1$.

If $z_0 = r_0 e^{i\theta}\ (0 < r_0 < 1)$ and $|W(z_0)| = \max_{|z| \leq r_0} |W(z)|$, then

i) $z_0 \frac{W'(z_0)}{W(z_0)} = m$ and

ii) $\mathrm{Re}\left(z_0 \frac{W''(z_0)}{W'(z_0)}\right) \geq m-1$

where m is real and

$$m \geq k\frac{|W(z_0)-a|^2}{|W(z_0)|^2-|a|^2} \geq k\frac{|W(z_0)|-|a|}{|W(z_0)|+|a|}$$

Applying the above lemma $D^{\alpha+p-1}f$ where $f \in M(p)$ we prove the following.

Theorem

Let $h(r,s,t)\in H(\alpha)$ and let $f(z)\in M(P)$ satisfy

i) $$\left(\frac{D^{\alpha+p}f(z)}{D^{\alpha+p-1}f(z)},\frac{D^{\alpha+p+1}f(z)}{D^{\alpha+p}f(z)},\frac{D^{\alpha+p+2}f(z)}{D^{\alpha+p+1}f(z)}\right)\in D\subset C^3 \text{ and}$$

ii) $$\left|h\left(\frac{D^{\alpha+p}f(z)}{D^{\alpha+p-1}f(z)},\frac{D^{\alpha+p+1}f(z)}{D^{\alpha+p}f(z)},\frac{D^{\alpha+p+2}f(z)}{D^{\alpha+p+1}f(z)}\right)\right|<J$$

for some α and J such that $\alpha\geq -p+1$, $J>1$ for all $z\in E_0$.
Then we have

$$\left|\frac{D^{\alpha+p}f(z)}{D^{\alpha+p-1}f(z)}\right|<J,\quad z\in E_0$$

Proof

We define the function $\omega(z)$ in E_0 by

$$\frac{D^{\alpha+p}f(z)}{D^{\alpha+p-1}f(z)}=\omega(z)\quad(\alpha\geq -p+1)$$

for $f(z)\in M(p)$. Then it follows that $\omega(z)$ is either analytic or meromorphic in E_0 and $\omega(0)=1$ and $\omega(z)\equiv 1$. With the aid of the following easily proved identity

$$z(D^{\alpha+p-1}f(z))'=(\alpha+p)D^{\alpha+p}f(z)-(\alpha+2p)D^{\alpha+p-1}f(z)$$

We compute

$$\frac{D^{\alpha+p+1}f(z)}{D^{\alpha+p}f(z)}=\frac{1}{(\alpha+p+1)}\left[1+(\alpha+p)\omega(z)+z\frac{\omega'(z)}{\omega(z)}\right] \quad \textit{and}$$

$$\frac{D^{\alpha+p+2}f(z)}{D^{\alpha+p+1}f(z)} = \frac{1}{(\alpha+p+2)}\left[2+(\alpha+p)\omega(z)+z\frac{\omega'(z)}{\omega(z)}+\right.$$

$$+(\alpha+p)z\omega'(z)+[z\omega'(z)/\omega(z)]+$$

$$\left.+\frac{z^2\omega''(z)/\omega(z)-(z\omega'(z)/\omega(z))^2}{1+(\alpha+p)\omega(z)+z\omega'(z)/\omega(z)}\right]$$

Suppose, if possible, that $z_0 = r_0 e^{i\theta}$ $(0<r_0<1)$ and

$|\omega(z_0)| = \max_{|z|\le r_0} |\omega(z)| = J$.

Letting $\omega(z_0) = Je^{i\theta}$ and using the lemma with $a=1$ and $k=1$ we get

$$\frac{D^{\alpha+p+1}f(z_0)}{D^{\alpha+p}f(z_0)} = \frac{1}{(\alpha+p+1)}[1+m+(\alpha+p)mJe^{i\theta}]$$

$$\frac{D^{\alpha+p+2}f(z_0)}{D^{\alpha+p+1}f(z_0)} = \frac{1}{(\alpha+p+2)}\times$$

$$\times\left[2+m+(\alpha+p)mJe^{i\theta}+\frac{m-m^2+(\alpha+p)mJe^{i\theta}+L}{1+m+(\alpha+p)Je^{i\theta}}\right]$$

Where $L = z_0\omega''(z_0)/\omega(z_0)$ and $m \ge (J-1)/(J+1)$.

Further, an application of (ii) in the lemma gives $\operatorname{Re} L \ge m(m-1)$.

Since $h(r,s,t)\in H(\alpha)$, we have

$$\left|h\left(\frac{D^{\alpha+p}f(z_0)}{D^{\alpha+p-1}f(z_0)},\frac{D^{\alpha+p+1}f(z_0)}{D^{\alpha+p}f(z_0)},\frac{D^{\alpha+p+2}f(z_0)}{D^{\alpha+p+1}f(z_0)}\right)\right| =$$

$$\left.\frac{(2+m+(\alpha+p)Je^{i\theta}+m-m^2+(\alpha+P)mJe^{i\theta}+L}{(\alpha+p+2)(1+m+(\alpha+p)Je^{i\theta}}\right)\right| \ge J$$

Which contradicts hypothesis (ii) of the theorem.
Therefore we conclude that

$$\omega(z)=\left|\frac{D^{a+p}f(z)}{D^{a+p-1}f(z)}\right|<J \text{ for all } z\in E_0$$

This proves the theorem.

References

‡ A.M.S. Subject Classification (2000) Primary: 30C 45, secondary 34C 40.

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